Exercise Set 3:
Cohomology Sets

Math 582e, Winter 2010, University of Washington

Due Wednesday, January 27, 2010

By "noncommutative" below I mean "not necessarily commutative".

1. Let G be a group and let $A = G$ equipped with its conjugation action, which is a noncummative G-module. Prove that $\mathrm{H}^0(G, A)$ is the center of G.
2. Let G be a group and let A be a noncommutative G-module. Is $\mathrm{H}^0(G, A)$ necessarily abelian?
3. Let $n \geq 2$ and let $G = \mathrm{GL}_n(\mathbb{C})$ be the group of invertible complex $n \times n$ matrices. Let $A = G$ equipped with its conjugation action, which is a noncummutative G-module. Prove that $\mathrm{H}^1(G, G)$ is infinite as follows.[1]
   (a) Let $Z(G)$ denote the center of G. Show that we have an exact sequence
   $$0 \to Z(G) \to G \to G/Z(G) \to 0$$
   of noncommutative G-modules.
   (b) Write down the long exact sequence (H^0 and H^1's) associated to the above short exact sequence.
   (c) Prove that $\mathrm{H}^0(G, G/Z(G)) = 0$, from the definition and what you (hopefully) know from linear algebra.
   (d) Prove that $\mathrm{H}^0(G, Z(G)) = \mathrm{Hom}(G, Z(G))$.
   (e) Deduce that we have an injective map of pointed sets
   $$\mathrm{Hom}(G, Z(G)) \hookrightarrow \mathrm{H}^1(G, G).$$
   (f) Prove that $\mathrm{Hom}(G, Z(G))$ is infinite by showing that any power of $\det : G \to \mathbb{C}$ defines an element of $\mathrm{Hom}(G, Z(G))$.

[1] I hope this is right: I just came up with this.