Exercise Set 2:
Morphisms of Pairs

Math 582e, Winter 2010, University of Washington

Due Friday (!), January 22, 2010

1. In Atiyah-Wall's article on page 98 of Cassels-Frohlich (and unfortunately my lecture!), one finds the following statement: "Let G' be a subgroup of G. If A' is a G'-module, we can form the G-module $A = \mathrm{Hom}_G(\mathbb{Z}[G], A')$: A is really a right G-module, but we turn it into a left G-module via: if $\varphi \in A$, then $g.\varphi$ is the homomorphism $g' \mapsto \varphi(g'g^{-1})$." However, yesterday Kevin Buzzard emailed me and said:

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Well I am a bit anti-Atiyah--Wall today. I spent over an hour trying
to work out why a diagram which should have commuted didn't commute,
and it was because of their erroneous assertion about the G-action
on the co-induced module [on page 98]. That was one of the few bits
they didn't copy out of Serre and they got it wrong!
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   What is the correct G-action? Prove your claim carefully.

2. Suppose $H \lhd G$. Show that we obtain an action of G/H on $\mathrm{H}^q(H, A)$ for all q induced by the conjugation action of G on H. [Hint: Use the morphism of pairs induced by conjugation by $t \in G$ on H and t^{-1} on A.]

3. Suppose E is an elliptic curve defined over $\mathbb{Q}$. Let p be a prime number such that the mod-p Galois representation

$$\overline{\rho}_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E[p])$$

   is surjective. Let K be the extension of $\mathbb{Q}$ obtained by adjoining all coordinates of the p-torsion points on E to $\mathbb{Q}$. Let L be any finite Galois extension of $\mathbb{Q}$ that contains K such that $p \nmid [L : K]$. Prove that

$$\mathrm{H}^1(\mathrm{Gal}(K/\mathbb{Q}), E(K)[p]) = 0,$$

   and that

$$\mathrm{H}^1(\mathrm{Gal}(L/\mathbb{Q}), E(L)[p]) \hookrightarrow \mathrm{Hom}(\mathrm{Gal}(L/K), E(L)[p]) = 0.$$