Modular Forms and Jacobi's Four Squares Theorem

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1 Introduction

For my final project I wanted to learn some more about modular forms. So I picked up Diamond and Shurman's *A First Course in Modular Forms* [3] and worked through some of the exercises. This paper strings together exercises and material from Sections 1.1 and 1.2 of that book to prove Jacobi's Four Squares Theorem. Though my primary goal was to get my hands dirty by working out the details of these exercises, I recognize that the reader may not find these details quite as fascinating. With that in mind, I have tried to omit some of the more straightforward calculations.

To facilitate the proof, I have freely assumed some results which appear in Diamond and Shurman either in the text or as exercises. These results are listed as facts in this paper.

2 Jacobi's Four Squares Theorem

In 1770, Lagrange proved the following result [2]:

Theorem 2.1. ***Lagrange's Four Squares Theorem**: Let $n \in \mathbb{N}$. Then there exist $a, b, c, d \in \mathbb{Z}$ such that*

$$n = a^2 + b^2 + c^2 + d^2.$$

For example, we have

$$\begin{aligned} 6 &= 0^2 + 1^2 + 1^2 + 2^2 \\ &= 0 + 1 + 1 + 4. \end{aligned}$$

One can generalize this idea and ask when can a natural number n be written as the sum of k integer squares. Or, more specifically, in how many ways can one write n as the sum of k squares. To that end, define the function:

$$r(n, k) = \#\{v \in \mathbb{Z}^k : v_1^2 + v_2^2 + \cdots + v_k^2\}.$$

We call $r(n, k)$ the *representation number of n by k squares*. Using this notation, we can easily state

Theorem 2.2. ***Jacobi's Four Squares Theorem**:*

$$r(n, k) \;=\; 8 \sum_{\substack{0 < d \mid n \\ 4 \nmid d}} d$$

The goal of this paper is to prove this result, first proven in 1834 by Jabobi [1]. To determine what representations are being counted, let us look at an example. Using the formula we compute

$$r(6, 4) = 8(1 + 2 + 3 + 6) = 96.$$

Comparing with our work above, we see that the formula counts different orderings separately. Similarly, $0^2 + 1^2 + 1^2 + 2^2$ and $0^2 + (-1)^2 + 1^2 + 2^2$ are regarded as distinct representations.

3 Modular Forms Background

Before tackling Jacobi's Four Squares Theorem, it will be useful to state a few definitons. The *modular group* $\mathrm{SL}_2(\mathbb{Z})$ consists of all integer matrices whose determinant is 1:

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) : ad - bc = 1 \right\}.$$

One can show that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$\alpha' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha'' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The *upper half plane* $\mathcal{H}$ consists of all points in the complex plane with positive imaginary part:

$$\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

We define the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ as follows:

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \tau \in \mathcal{H},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Let $k \in \mathbb{N}$. A function $f : \mathcal{H} \to \mathbb{C}$ is said to be *weakly modular of weight k* if for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \qquad \tau \in \mathcal{H},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To simplify notation, for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, define the *factor of automorphy* $j : \mathrm{SL}_2(\mathbb{Z}) \times \mathcal{H} \to \mathbb{C}$:

$$j(\gamma, \tau) = (c\tau + d), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

And now, for $k \in \mathbb{N}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, define the *weight k operator* $[\gamma]_k$ on functions $f : \mathcal{H} \to \mathbb{C}$ by

$$(f[\gamma]_k)(\tau) = j(\gamma, \tau)^{-k} f(\tau).$$

We say a function $f : \mathcal{H} \to \mathbb{C}$ is *holomorphic at ∞* if f has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \qquad e^{2\pi i\tau}.$$

To show a function f is holomporphic at ∞, it suffices to show that $f(\tau)$ is bounded as $\Im(\tau) \to \infty$.

Definition 3.1. *A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight k if*

1. *f is holomorpic*
2. *f is weakly modular of weight k*
3. *f is holomorphic at ∞*

We denote the set of modular forms of weight k by $\mathcal{M}_2(\Gamma_0(N))$

There are also other, more general modular forms. In order to define them, we need the notion of a congruence subgroup: The *principal congruence subgroup of level N* is given by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

And a *congruence subgroup of level N* is a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma(N) \subseteq \Gamma$. For example

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

is a congruence subgroup of level N. We will be particularly interested in the congruence subgroup $\Gamma_0(4)$.

Proposition 3.2. *The congruence subgroup $\Gamma_0(4)$ is generated by the matrices $\pm\gamma'$ and $\pm\gamma''$ where*

$$\gamma' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \gamma'' = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}.$$

Proof: Clearly, $\gamma', \gamma'' \in \Gamma_0(4)$ so let $\gamma_1 \in \Gamma_0(4)$ be given by

$$\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

Note that, since $a_1d_1 - b_1c_1 = 1$ and $c_1 \equiv 0 \pmod 4$, it must be the case that d_1 is odd. Thus, if $c_1 \neq 0$, there exists $n_1 \in \mathbb{Z}$ such that $|n_1c_1 + d_1| < |c_1|/2$. Now, consider the matrix

$$\gamma_2 = \gamma_1(\gamma')^{n_1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & n_1a_1 + b_1 \\ c_1 & n_1c_1 + d_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Note that d_2 is also odd. Therefore, since $|d_2| < |c_2|/2$ and $c_2 \equiv 0 \pmod 4$, there exists $n_2 \in \mathbb{Z}$ such that $|c_2 + 4n_2d_2| < |d_2|$. So consider the matrix $\gamma_3 = \gamma_2(\gamma'')^{n_2}$. Continuing in this fashion, we obtain a decreasing sequence of positive integers $|d_2|, |c_3|, |d_4|, |c_5|, \ldots$ which must therefore terminate. In particular, since d_n is odd for all $n \in \mathbb{N}$, by repeatedly mutlitplying γ_1 on the left by powers of γ' and γ'', we obtain a matrix $\gamma \in \Gamma_0(4)$ of the form

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Since $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have $ad = 1$ and, therefore,

$$\gamma = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} = (\pm\gamma')(\gamma')^{-1}(\gamma')^b.$$

So $\gamma \in \langle \pm\gamma', \pm\gamma'' \rangle$, as desired.

Definition 3.3. *A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight k with respect to the congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ if*

1. *f is holomorpic*
2. *f is weakly modular of weight k with respect to Γ: i.e., $(f[\gamma]_k)(\tau) = f(\tau)$ for all $\tau \in \mathcal{H}$ and $\gamma \in \Gamma$*
3. *$f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*

We denote the set of all such modular forms by $\mathcal{M}_k(\Gamma)$.

Fact 3.4. *Let $f : \mathcal{H} \to \mathbb{C}$ be a function which satisfies conditions 1 and 2 in Definition 3.3 for some congruence subgroup $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$ of level N. And, furthermore, suppose f has a Fourier expansion*

$$f(\tau) = \sum_{n=0}^{\infty} a_n q_N^n, \qquad q_N = e^{2\pi i\tau/N},$$

where $|a_n| \leq Cn^r$ for some fixed positive constants C and r. Then $f \in \mathcal{M}_k(\Gamma)$.

4 The Eisenstein Series of Weight 2

Let $k > 2$ be even. The (non-normalized) *Eisenstein series of weight k*, G_k, is given by

$$G_k(\tau) = \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(c\tau + d)^k}$$

Proposition 4.1. *G_k is a modular form of weight k.*

A proof of Proposition 4.1 is included in the appendix and relies on the fact that G_k converges absolutely for all $\tau \in \mathcal{H}$. One can also define the Eisenstein series of weight 2:

$$G_2(\tau) = \sum_{c\in\mathbb{Z}} \sum_{d\in\mathbb{Z}'_c} \frac{1}{(c\tau+d)^2}$$

where $\mathbb{Z}'_c$ is given by

$$\mathbb{Z}'_c = \begin{cases} \mathbb{Z} & \text{if } c \neq 0, \\ \mathbb{Z}\backslash\{0\} & \text{if } c = 0. \end{cases}$$

G_2 does not converge absolutely, but does converge conditionally and in fact

Proposition 4.2.

$$G_2(\tau) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d \right) q^n, \qquad q = e^{2\pi\tau},$$

where ζ is the Riemann zeta function

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

To prove this, we will require the following result:

Fact 4.3. *For every $\tau \in \mathcal{H}$,*

$$\frac{1}{\tau} + \sum_{d=1}^{\infty} \left(\frac{1}{\tau-d} + \frac{1}{\tau+d} \right) = \pi\cot(\pi\tau) = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m, \qquad q = e^{2\pi i\tau}. \tag{1}$$

Proof of Proposition 4.2: Differentiating the left and right terms in the Equation 1, we obtain

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{1}{\tau} + \sum_{d=1}^{\infty} \left(\frac{1}{\tau-d} + \frac{1}{\tau+d} \right)\right) &= \frac{-1}{\tau^2} + \sum_{d=1}^{\infty} \left(\frac{-1}{(\tau-d)^2} + \frac{-1}{(\tau+d)^2} \right) \\ &= -\sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^2}, \end{aligned}$$

and (keeping in mind that $q = e^{2\pi i}$),

$$\begin{aligned} \frac{d}{d\tau}\left(\pi i - 2\pi i \sum_{m=0}^{\infty} q^m \right) &= -2\pi i \sum_{m=0}^{\infty} (2\pi i) m q^m \\ &= 4\pi^2 \sum_{m=1}^{\infty} m q^m. \end{aligned}$$

Thus,

$$\begin{aligned} G_2(\tau) &= \sum_{d\in\mathbb{Z}'_0} \frac{1}{(d)^2} + 2\sum_{c=1}^{\infty} \sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^2} \\ &= 2\zeta(2) + 2\sum_{c=1}^{\infty} \left(-4\pi^2 \sum_{m=1}^{\infty} m(q^c)^m \right) \\ &= 2\zeta(2) - 8\pi^2 \sum_{c=1}^{\infty} \sum_{m=1}^{\infty} m q^{cm} \end{aligned}$$

$$G_2(\tau) \quad = \quad 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d \right) q^n.$$

We may now show that G_2 is holomorphic on $\mathcal{H}$ and at ∞: Using induction on the number of (not necessarily distinct) prime factors of n, one can show that

$$\sum_{0<d|n} d < n^2.$$

Let $0 < A < 1$. And let $\Omega_A \subseteq \mathcal{H}$ be the region consisting of all points with imaginary part greater than $-\log(A)/(2\pi)$. Then, for all $\tau \in \Omega_A$,

$$|q| = |e^{2\pi i \tau}| = e^{-2\pi \Im(\tau)} < A.$$

Note that for all $\tau \in \Omega_A$, since $2A/(A+1) < 1$ and exponential growth dominates polynomial growth,

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\sum_{0<d|n} dq^n}{\left(\frac{A+1}{2}\right)^n} \right| \quad &\leq \quad \lim_{n\to\infty} \left| \frac{n^2 A^n}{\left(\frac{A+1}{2}\right)^n} \right| \\ &= \quad \lim_{n\to\infty} \left| n^2 \left(\frac{2A}{A+1} \right)^n \right| \\ &= \quad 0. \end{aligned}$$

The geometric series $\sum_{n=0}^{\infty} \left(\frac{2A}{A+1}\right)^n$ converges. So, by the limit comparison test,

$$-8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d \right) q^n$$

converges absolutely and uniformly on Ω_A. Every compact set $K \subseteq \mathcal{H}$ is contained in a set Ω_A for some A. So by the Weierstrass convergence theorem (see for example [4]), G_2 is holomorphic on $\mathcal{H}$. Furthermore, given a sequence $\{\tau_m\}_{m=1}^{\infty} \subseteq \mathcal{H}$ satisfying $\lim_{m\to\infty} \Im(\tau_m) = \infty$, it must be the case that $\tau_m \in \Omega_{1/2}$ for all m greater than some fixed M. Thus, for all $m \geq M$,

$$\sum_{n=1}^{\infty} \left| \left(\sum_{0<d|n} d \right) e^{2\pi i \tau_m} \right| \quad \leq \quad \sum_{n=1}^{\infty} n^2 \left(\frac{1}{2} \right)^n$$

which we know converges from the limit comparison test above. So $G_2(\tau)$ is bounded as $\Im(\tau) \to \infty$ and, therefore, is holomorphic at ∞. However, G_2 is not a modular form. Instead,

Proposition 4.4. *For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,*

$$(G_2[\gamma]_2)(\tau) = G_2(\tau) - \frac{2\pi i c}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2}$$

Proof: Let $\gamma, \gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ which satisfy Equation 2. Using the relations

$$\begin{aligned} (G_2[\gamma_1\gamma_2]_2)(\tau) \quad &= \quad j(\gamma_2, \tau)^{-2} j(\gamma_1, \gamma_2\tau)^{-2} G(\gamma_1(\gamma_2\tau)) \quad \text{and} \\ G_2(\tau) \quad &= \quad G_2(\gamma\gamma^{-1}\tau), \end{aligned}$$

it is relatively straightforward to show that Equation 2 is also satisfied by $\gamma_1\gamma_2$ and γ^{-1}. Let us therefore simply check that Equation 2 is satisfied by the generators α' and α'' of $\mathrm{SL}_2(\mathbb{Z})$. By Proposition 4.2,

$$\begin{aligned}(G_2[\alpha']_2)(\tau) &= G_2(\tau+1)\\ &= 2\zeta(2) - 8\pi^2\sum_{n=1}^{\infty}\left(\sum_{0<d|n} d\right)e^{2\pi i(\tau+1)n}\\ &= 2\zeta(2) - 8\pi^2\sum_{n=1}^{\infty}\left(\sum_{0<d|n} d\right)e^{2\pi i\tau n}\\ &= G_2(\tau).\end{aligned}$$

Showing the result for α'' is a bit trickier:

$$\begin{aligned}(G_2[\alpha'']_2)(\tau) &= \frac{1}{\tau^2}G_2\left(\frac{-1}{\tau}\right)\\ &= \frac{1}{\tau^2}\sum_{c\in\mathbb{Z}}\sum_{d\in\mathbb{Z}'_c}\frac{1}{(c\frac{-1}{\tau}+d)^2}\\ &= \sum_{c\in\mathbb{Z}}\sum_{d\in\mathbb{Z}'_c}\frac{1}{(d\tau-c)^2}\\ &= 2\zeta(2)+\sum_{c\in\mathbb{Z}}\sum_{d\in\mathbb{Z}\backslash\{0\}}\frac{1}{(d\tau-c)^2}\end{aligned}$$

By relabeling, we obtain:

$$(G_2[\alpha'']_2)(\tau) = 2\zeta(2)+\sum_{d\in\mathbb{Z}}\sum_{c\in\mathbb{Z}\backslash\{0\}}\frac{1}{(c\tau+d)^2} \tag{3}$$

Note that for each $c \in \mathbb{Z}\backslash\{0\}$,

$$\begin{aligned}\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)(c\tau+d+1)} &= \lim_{n\to\infty}\sum_{d=-n}^{n}\frac{1}{(c\tau+d)(c\tau+d+1)}\\ &= \lim_{n\to\infty}\sum_{d=-n}^{n}\left(\frac{1}{(c\tau+d)}-\frac{1}{(c\tau+d+1)}\right)\\ &= \lim_{n\to\infty}\frac{1}{(c\tau-n)}-\frac{1}{(c\tau+n+1)}\\ &= 0.\end{aligned}$$

Thus,

$$\sum_{c\in\mathbb{Z}\backslash\{0\}}\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)(c\tau+d+1)} = 0.$$

and, therefore, we may subtract this sum from $G_2(\tau)$:

$$\begin{aligned}G_2(\tau) &= G_2(\tau)-\sum_{c\in\mathbb{Z}\backslash\{0\}}\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)(c\tau+d+1)}\\ &= 2\zeta(2)+\sum_{c\in\mathbb{Z}\backslash\{0\}}\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)^2} - \sum_{c\in\mathbb{Z}\backslash\{0\}}\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)(c\tau+d+1)}\end{aligned}$$

$$\begin{aligned} G_2(\tau) &= 2\zeta(2) + \sum_{c\in\mathbb{Z}\backslash\{0\}} \sum_{d\in\mathbb{Z}} \left(\frac{1}{(c\tau+d)^2} - \frac{1}{(c\tau+d)(c\tau+d+1)} \right) \\ &= 2\zeta(2) + \sum_{c\in\mathbb{Z}\backslash\{0\}} \sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^2(c\tau+d+1)} \end{aligned}$$

Note that for all $c \in \mathbb{Z}\backslash\{0\}$ and $d \in \mathbb{Z}$,

$$\left| \frac{(c\tau+d)^2(c\tau+d+1)}{(c\tau+d)^3} \right| = \left| \frac{c\tau+d+1}{c\tau+d} \right| = 1 + \left| \frac{1}{c\tau+d} \right| \leq 1 + \frac{1}{\Im(\tau)}$$

And, by the same argument as in the proof of Proposition 4.1,

$$\sum_{c\in\mathbb{Z}\backslash\{0\}} \sum_{d\in\mathbb{Z}} \frac{1}{|c\tau+d|^3}$$

converges. So by the limit comparison test,

$$\sum_{c\in\mathbb{Z}\backslash\{0\}} \sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^2(c\tau+d+1)}$$

converges absolutely. We may therefore change the order of summation. Hence,

$$G_2(\tau) = 2\zeta(2) + \sum_{d\in\mathbb{Z}} \sum_{c\in\mathbb{Z}\backslash\{0\}} \frac{1}{(c\tau+d)^2(c\tau+d+1)}$$

From Equation 3, we have

$$\begin{aligned} G_2(\tau) &= 2\zeta(2) + \sum_{d\in\mathbb{Z}} \sum_{c\in\mathbb{Z}\backslash\{0\}} \frac{1}{(c\tau+d)^2(c\tau+d+1)} + (G_2[\alpha'']_2)(\tau) - 2\zeta(2) - \sum_{d\in\mathbb{Z}} \sum_{c\in\mathbb{Z}\backslash\{0\}} \frac{1}{(c\tau+d)^2} \\ &= (G_2[\alpha'']_2)(\tau) + \sum_{d\in\mathbb{Z}} \sum_{c\in\mathbb{Z}\backslash\{0\}} \left(\frac{1}{(c\tau+d)^2(c\tau+d+1)} - \frac{1}{(c\tau+d)^2} \right) \\ &= (G_2[\alpha'']_2)(\tau) - \sum_{d\in\mathbb{Z}} \sum_{c\in\mathbb{Z}\backslash\{0\}} \frac{1}{(c\tau+d)(c\tau+d+1)} \\ &= (G_2[\alpha'']_2)(\tau) - \sum_{d\in\mathbb{Z}} \sum_{c\in\mathbb{Z}\backslash\{0\}} \left(\frac{1}{(c\tau+d)} - \frac{1}{(c\tau+d+1)} \right) \end{aligned}$$

Note that

$$\begin{aligned} \sum_{d\in\mathbb{Z}} \sum_{c\in\mathbb{Z}\backslash\{0\}} \left(\frac{1}{(c\tau+d)} - \frac{1}{(c\tau+d+1)} \right) &= \lim_{n\to\infty} \sum_{d=-n}^{n-1} \sum_{c\in\mathbb{Z}\backslash\{0\}} \left(\frac{1}{(c\tau+d)} - \frac{1}{(c\tau+d+1)} \right) \\ &= \lim_{n\to\infty} \sum_{c\in\mathbb{Z}\backslash\{0\}} \sum_{d=-n}^{n-1} \left(\frac{1}{(c\tau+d)} - \frac{1}{(c\tau+d+1)} \right) \\ &= \lim_{n\to\infty} \sum_{c\in\mathbb{Z}\backslash\{0\}} \left(\frac{1}{(c\tau-n)} - \frac{1}{(c\tau+n)} \right) \\ &= \lim_{n\to\infty} \sum_{c\in\mathbb{Z}\backslash\{0\}} -\frac{1}{\tau} \left(\frac{1}{(n/\tau-c)} + \frac{1}{(n/\tau+c)} \right) \end{aligned}$$

Adding 0 in order to apply Fact 4.3, and keeping in mind that $\Im(\tau) > 0$, we obtain

$$\begin{aligned}
\sum_{d\in\mathbb{Z}}\sum_{c\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{(c\tau+d)}-\frac{1}{(c\tau+d+1)}\right) &= \lim_{n\to\infty} 2\frac{1}{\tau}\frac{1}{n/\tau}-2\frac{1}{\tau}\frac{1}{n/\tau}-2\frac{1}{\tau}\sum_{c=1}^{\infty}\left(\frac{1}{(n/\tau-c)}+\frac{1}{(n/\tau+c)}\right)\\
&= \lim_{n\to\infty}\frac{2}{n}-2\frac{1}{\tau}\pi\cot\left(\pi\frac{n}{\tau}\right)\\
&= \lim_{n\to\infty}-\frac{2\pi i}{\tau}\frac{e^{\pi in/\tau}+e^{-\pi in/\tau}}{e^{\pi in/\tau}-e^{-\pi in/\tau}}\\
&= \lim_{n\to\infty}-\frac{2\pi i}{\tau}\frac{1+e^{-2\pi in/\tau}}{1-e^{-2\pi in/\tau}}\\
&= -\frac{2\pi i}{\tau}
\end{aligned}$$

Therefore,

$$G_2(\tau) = (G_2[\alpha'']_2)(\tau) - \frac{2\pi i}{\tau} \qquad \text{or} \qquad (G_2[\alpha'']_2)(\tau) = G_2(\tau) - \frac{2\pi i}{-\tau}$$

This completes the proof.

Although, G_2 is not modular with respect to any congruence subgroup, we can use it to define functions which are. For each $N \in \mathbb{N}$, let $G_{2,N} : \mathcal{H} \to \mathbb{C}$ be given by

$$G_{2,N}(\tau) = G_2(\tau) - NG_2(N\tau).$$

Proposition 4.5. $G_{2,N} \in \mathcal{M}_2(\Gamma_0(N))$

Proof: Let $\gamma \in \Gamma_0(N)$ be given by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

And let $\tilde{\gamma}$ be given by

$$\tilde{\gamma} = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1/N & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & Nb \\ c/N & d \end{pmatrix}.$$

Note that $\tilde{\gamma}$ is in $\mathrm{SL}_2(\mathbb{Z})$ and that for all $\tau \in \mathcal{H}$, $\tilde{\gamma}N\tau = N\gamma\tau$. Furthermore,

$$j(\tilde{\gamma}, N\tau) = \frac{c}{N}(N\tau) + d = c\tau + d = j(\gamma, \tau).$$

Thus:

$$\begin{aligned}
(G_{2,N}[\gamma]_2)(\tau) &= j(\gamma,\tau)^{-2}G_{2,N}(\gamma\tau)\\
&= j(\gamma,\tau)^{-2}\left(G_2(\gamma\tau) - NG_2(N\gamma\tau)\right)\\
&= j(\gamma,\tau)^{-2}G_2(\gamma\tau) - Nj(\tilde{\gamma},N\tau)^{-2}G_2(\tilde{\gamma}N\tau)\\
&= (G_2[\gamma]_2)(\tau) - N(G_2[\tilde{\gamma}]_2)(N\tau)\\
&= G_2(\tau) - \frac{2\pi ic}{c\tau+d} - N\left(G_2(N\tau) - \frac{2\pi i(c/N)}{(c/N)(N\tau)+d}\right)\\
&= G_2(\tau) - NG_2(N\tau)\\
&= G_{2,N}(\tau).
\end{aligned}$$

So $G_{2,N}$ is weakly modular of weight 2 with respect to $\Gamma_0(N)$. Since G_2 is holomorphic on $\mathcal{H}$ so is $G_{2,N}$. Now, note that, by Proposition 4.2,

$$\begin{aligned}
G_{2,N}(\tau) &= 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d \right) q^n - N \left(2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d \right) q^{Nn} \right) \\
&= -2(N-1)\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d \right) q^n + 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} Nd \right) q^{Nn} \\
&= -(N-1)\frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d \right) q^n + 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{\substack{0<d|Nn \\ N|d}} d \right) q^{Nn} \qquad (4) \\
&= -(N-1)\frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{\substack{0<d|n \\ N\nmid d}} d \right) q^n.
\end{aligned}$$

(Here we used the identity $\zeta(2) = \pi^2/6$.) Or, alternatively, letting $q_N = e^{2\pi i\tau/N}$, we obtain

$$\begin{aligned}
G_{2,N}(\tau) &= -(N-1)\frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{\substack{0<d|n \\ N\nmid d}} d \right) q_N^{Nn} \\
&= -(N-1)\frac{\pi^2}{3} - 8\pi^2 \sum_{\substack{n>0 \\ N|n}} \left(\sum_{\substack{0<d|n/N \\ N\nmid d}} d \right) q_N^{n}.
\end{aligned}$$

Recall that

$$\sum_{\substack{0<d|n/N \\ N\nmid d}} d < n^2$$

for all n. Thus, by Fact 3.4, we conclude that $G_{2,N}$ is a modular form of weight 2 with respect to $\Gamma_0(N)$.

Fact 4.6. *$\mathcal{M}_2(\Gamma_0(4))$ is a 2-dimensional vector space over $\mathbb{C}$.*

Using Equation 4, we can write out the Fourier expansions of $G_{2,2}$ and $G_{2,4}$:

$$\begin{aligned}
G_{2,2}(\tau) &= -\frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{\substack{0<d|n \\ 2\nmid d}} d \right) q^n \\
&= -\frac{\pi^2}{3}\left(1 + 24q + 24q^2 + \ldots\right)
\end{aligned}$$

And,

$$\begin{aligned} G_{2,4}(\tau) &= -\pi^2 - 8\pi^2 \sum_{n=1}^{\infty} \left(\sum_{\substack{0\,<\,d|n \\ 4\,\nmid\, d}} d \right) q^n \\ &= -\pi^2 \left(1 + 8q + 24q^2 + ...\right) \end{aligned}$$

From this we see that $G_{2,2}$ and $G_{2,4}$ are independent. It follows that $G_{2,2}$ and $G_{2,4}$ form a basis for $\mathcal{M}_2(\Gamma_0(4))$.

5 Jacobi's Four Squares Theorem Revisited

Let us define the generating function $\theta : \mathcal{H} \times \mathbb{N} \to \mathbb{C}$:

$$\theta(\tau, k) = \sum_{n=0}^{\infty} r(n,k) q^n, \qquad q = e^{2\pi i \tau}$$

Fix k and note that $r(n,k) < 2^k n^k$ for all $n > 1$. Hence, one can show that $\theta(\tau, k)$ converges absolutely and uniformly on compact sets $K \subseteq \mathcal{H}$ in the same way we showed the Fourier series of G_2 converges. So $\theta(\tau, k)$ is holomorphic. By construction, for all $\tau \in \mathcal{H}$, we have

$$\theta(\tau + 1, k) = \theta(\tau, k). \tag{5}$$

Furthemore, we may take the Cauchy product of $\theta(\tau, k_1)$ and $\theta(\tau, k_2)$ to obtain the relation

$$\begin{aligned} \theta(\tau, k_1)\theta(\tau, k_2) &= \left(\sum_{n=0}^{\infty} r(n,k_1) q^n\right)\left(\sum_{n=0}^{\infty} r(n,k_2) q^n\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} r(\ell, k_1) r(n-\ell, k_2)\right) q^n. \end{aligned}$$

A little thought will convince one that

$$\sum_{\ell=0}^{n} r(\ell, k_1) r(n-\ell, k_2) = r(n, k_1 + k_2).$$

Hence

$$\begin{aligned} \theta(\tau, k_1)\theta(\tau, k_2) &= \sum_{n=0}^{\infty} r(n, k_1 + k_2) q^n \\ &= \theta(\tau, k_1 + k_2). \end{aligned}$$

So, in particular

$$\theta(\tau, k) = \theta(\tau, 1)^k. \tag{6}$$

Another important identity is given by

Fact 5.1.

$$\theta\left(-\frac{1}{4\tau}, 1\right) = \sqrt{-2i\tau}\,\theta(\tau, 1) \tag{7}$$

where the formula calls for the principal branch of the square root function.

Applying Equations 5 and 7, we obtain

$$\begin{aligned}
\theta\left(\frac{\tau}{4\tau+1},1\right) &= \theta\left(\frac{1}{-4(-1-1/(4\tau))},1\right) \\
&= \sqrt{-2i\left(-1-\frac{1}{4\tau}\right)}\;\theta\left(-1-\frac{1}{4\tau},1\right) \\
&= \sqrt{2i\frac{4\tau+1}{4\tau}}\;\theta\left(-\frac{1}{4\tau},1\right) \\
&= \sqrt{2i\frac{4\tau+1}{4\tau}}\sqrt{-2i\tau}\;\theta\left(\tau,1\right) \\
&= \sqrt{4\tau+1}\,\theta\left(\tau,1\right).
\end{aligned}$$

Applying Equation 6, we obtain

$$\theta\left(\frac{\tau}{4\tau+1},4\right) = (4\tau+1)^2\theta\left(\tau,4\right).$$

Combining this with Equation 5, we see that $\theta(\tau,4)$ is weakly modular of weight 2 with respect to $\Gamma_0(4)$. And, using Fact 3.4, one can show that $\theta(\tau,4) \in \mathcal{M}_2(\Gamma_0(4))$. Writing out the first few terms of $\theta(\tau,4)$, we obtain

$$\begin{aligned}
\theta(\tau,4) &= 1 + r(1,4)q + r(2,4)q^2 + ... \\
&= 1 + 8q + 24q^2 + ...
\end{aligned}$$

Comparing this with the Fourier series of $G_{2,2}$ and $G_{2,4}$ and using the fact that $G_{2,2}$ and $G_{2,4}$ form a basis for $\mathcal{M}_2(\Gamma_0(4))$, we see that

$$\theta(\tau,4) = -\frac{1}{\pi^2}G_{2,4}.$$

Hence,

$$r(n,4) = 8\sum_{\substack{0<d|n \\ 4\nmid d}} d.$$

This completes the proof of Jacobi's Four Squares Theorem.

6 Appendix

Proof of Proposition 4.1: To show $G_k \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$, we must show that the following three conditions are satisfied:

1. G_k is holomorphic
2. $(G_k[\gamma]_k)(\tau) = G_k(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$
3. G_k is holomorphic at ∞

Let A and B be positive real numbers and let $\Omega_{A,B} \subseteq \mathcal{H}$ consist of all points with real part between $-A$ and A and with imaginary part greater than B:

$$\Omega_{A,B} = \{z \in \mathcal{H} : |\Re(z)| < A, \Im(z) > B\}$$

And let $C_{A,B}$ be given by

$$C_{A,B} = \frac{AB}{2(1+A+B)^2}.$$

Let $\delta \in \mathbb{R}$ with $|\delta| \leq 2A$. Then, for all $\tau \in \Omega_{A,B}$,

$$\begin{aligned}
|\tau + \delta| &\geq \Im(\tau) \\
&> B \\
&> \sup\left\{ B\frac{A}{2(A+B+1)^2}, B\frac{A^2}{(A+B+1)^2} \right\} \\
&= C_{A,B}\sup\{1, 2A\} \\
&\geq C_{A,B}\sup\{1, |\delta|\}.
\end{aligned}$$

Similarly, if $|\delta| > 2A$, note that

$$\begin{aligned}
|\tau + \delta| &\geq \sup\{|\Re(\tau) + \delta|, \Im(\tau)\} \\
&\geq \sup\{|\delta| - A, B\} \\
&> \sup\left\{ \frac{|\delta|}{2}, B \right\} \\
&> \sup\left\{ \frac{|\delta|}{2}\frac{AB}{(A+B+1)^2}, B\frac{A}{2(A+B+1)^2} \right\} \\
&\geq C_{A,B}\sup\{|\delta|, 1\}.
\end{aligned}$$

So, in fact for all $\delta \in \mathbb{R}$, we have

$$|\tau + \delta| > C_{A,B}\sup\{1, |\delta|\}.$$

We may use this fact to show that G_k converges absolutely and uniformly on compact sets: Let $K \in \mathcal{H}$ be compact. Then, since K, is bounded $K \subseteq \Omega_{A,B}$ for some $A, B > 0$. Thus, for all $\tau \in K$,

$$\begin{aligned}
\sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{|c\tau + d|^k} &= 2\zeta(k) + \sum_{c\in\mathbb{Z}\setminus\{0\}}\sum_{d\in\mathbb{Z}} \frac{1}{|c\tau + d|^k} \\
&= 2\zeta(k) + \sum_{c\in\mathbb{Z}\setminus\{0\}}\sum_{d\in\mathbb{Z}} \frac{1}{(|c||\tau + d/c|)^k} \\
&< 2\zeta(k) + \sum_{c\in\mathbb{Z}\setminus\{0\}}\sum_{d\in\mathbb{Z}} \frac{1}{(|c|C_{A,B}\sup\{1, |d/c|\})^k} \\
&= 2\zeta(k) + \sum_{c\in\mathbb{Z}\setminus\{0\}}\sum_{d\in\mathbb{Z}} \frac{1}{(C_{A,B})^k(|c|\sup\{1, |d/c|\})^k} \\
&= 2\zeta(k) + \frac{1}{(C_{A,B})^k}\sum_{c\in\mathbb{Z}\setminus\{0\}}\sum_{d\in\mathbb{Z}} \frac{1}{(\sup\{|c|, |d|\})^k}
\end{aligned}$$

which converges for $k > 1$. Note that for each $n \in \mathbb{Z}_{\geq 1}$, there are $2(2n+1)$ points (c, d) in $\mathbb{Z}^2$ with $c = \pm n$. Similarly, there are $2(2n-1)$ points with $d = \pm n$ and $|c| < n$. So

$$\sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{|c\tau + d|^k} \leq 2\zeta(k) + \frac{1}{(C_{A,B})^k}\sum_{n=1}^{\infty}\frac{8n}{n^k}$$

which converges because $k > 2$. So G_k converges absolutely and uniformly on compact sets. Therefore, since each of the finite sums is holomorphic, by the Weierstrass convergence theorem, G_k is itself holomorphic. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be given by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$\begin{aligned}(G_k[\gamma]_k)(\tau) &= \frac{1}{(c\tau+d)^k}\sum_{(c',d')\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(c'(\gamma\tau)+d')^k}\\ &= \sum_{(c',d')\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(c\tau+d)^k\left(c'\left(\frac{a\tau+b}{c\tau+d}\right)+d'\right)^k}\\ &= \sum_{(c',d')\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{((c'a+d'c)\tau+(c'b+d'd))^k}\end{aligned}$$

Note that for any $n, m \in \mathbb{Z}$, if we set $c' = nd - mc$ and $d' = -nb + ma$, we have

$$\begin{aligned}(c'a+d'c)\tau+(c'b+d'd) &= ((nd-mc)a+(-nb+ma)c)\tau+((nd-mc)b+(-nb+ma)d)\\ &= (nad-nbc-mac+mac)\tau+(nbd-mbc+-nbd+mad)\\ &= n(ad-bc)\tau+m(ad-bc)\\ &= n\tau+m.\end{aligned}$$

Now, let $c', d', c'', d'' \in \mathbb{Z}$ such that

$$(c'a+d'c)\tau+(c'b+d'd) = (c''a+d''c)\tau+(c''b+d''d).$$

We wish to show that $c' = c''$ and $d' = d''$. Since $\Im(\tau) \neq 0$,

$$\begin{aligned}(c'-c'')a+(d'-d'')c = 0 \quad \text{and}\\ (c'-c'')b+(d'-d'')d = 0.\end{aligned}$$

Since $ad - bc = 1$, a and b cannot simultaneously be 0. Nor can a and c, b and d, or c and d. Thus, if any of a, b, c, or d is 0, we are done. If not, note that

$$\begin{aligned}d((c'-c'')a+(d'-d'')c)-c((c'-c'')b+(d'-d'')d) &= (ad-bc)(c'-c'')+(cd-cd)(d'-d'')\\ &= c'-c''\\ &= 0.\end{aligned}$$

So, as (c', d') ranges over $\mathbb{Z}^2$, so does $(c'a + d'c, c'b + d'd)$ with $(c'a + d'c, c'b + d'd) = (0, 0)$ exactly when $(c', d') = (0, 0)$. Hence:

$$\sum_{(c',d')\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{((c'a+d'c)\tau+(c'b+d'd))^k} = \sum_{(c'',d'')\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(c''\tau+d'')^k}$$

Therefore

$$(G_k[\gamma]_k)(\tau) = G_k(\tau).$$

So G_k is weakly modular of weight k. In particular, $G_k(\tau + n) = G_k(\tau)$ for all $n \in \mathbb{Z}$. It only remains to show that G_k is holomorphic at ∞. Let $\{\tau_j\}_{j=1}^{\infty} \subseteq \mathcal{H}$ be a sequence of points satisfying $\Im(\tau_j) \to \infty$. Then there exists some $M \in \mathbb{N}$ such that for all $m \geq M$, $\Im(\tau_m) > 1$. Note that for each τ_m, there exists $\ell_m \in \mathbb{Z}$ such that $\tau_m + \ell_m \in \Omega_{1,1}$. Thus, for all $m \geq M$,

$$|G_k(\tau_m)| \;=\; |G_k(\tau_m+\ell_m)| \;<\; 2\zeta(k)+\frac{8}{(C_{1,1})^k}\sum_{n=1}^{\infty}\frac{1}{n^{k-1}} \;<\; \infty$$

Hence, $G_k(\tau)$ is bounded as $\Im(\tau) \to \infty$ and, therefore, holomorphic at ∞. This completes the proof.

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