Orders of Quaternion Algebras Over Number Fields

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One generalization of modular forms, Hilbert modular forms, are currently popular to study. The methods used to compute Hilbert modular forms exploit the Jacquet-Langlands correspondence with spaces of quaternionic modular forms. The goal of this paper, is to give someone with a back ground of a basic algebraic number theory course plus some modular forms enough background in quaternion algebras to understand the statement of the Jacquet-Langlands correspondence.

This paper draws mostly from the work of John Voight in *The arithmetic of quaternion algebras* and Lassina Dembele and John Voight in *Explicit methods for Hilbert modular forms.*

1 Quaternion Algebras over Fields

An algebra over the field F is a ring B equipped with an embedding $F \hookrightarrow B$ such that the image of F lies in the center of B. We then identify F with its image in B. Further, we say the dimension of B is the dimension $\dim_F B$ of B as an F-vector space.

A quaternion algebra over a field F is an F-algebra $B = \left(\frac{a,b}{F}\right)$ with basis $1, i, j, ij$ where if $\mathrm{char}(F) \neq 2$

$$i^2 = a, j^2 = b, \text{ and } ji = -ij$$

and $a, b \in F^\times$ and if $\mathrm{char}(F) = 2$,

$$i^2 + i = a, j^2 = b, ji = (i+1)j$$

and $a \in F$, $b \in F^\times$. Or equivalently, a quaternion algebra B is a central simple F-algebra with $\dim_F B = 4$.

Another useful way to view quaternion algebra is as an algebra containing two quadratic extension of F which anti-commute. Say K is a quadratic extension of F contained in B. Then $B = \left(\frac{a,b}{F}\right)$ where $i^2 = a$ and $i \subset K$ and $ij = -ji$, i.e., we can view $K = F[i] = F \oplus Fi$ as an F-algebra generated by i. Then either $K = F(\sqrt{a})$ or K has a zero divisor. If K is a field, then the notation for this quaternion algebra is $B = \left(\frac{K,b}{F}\right) \subset M_2(K)$. If K has a zero divisor, then $B \cong M_2(F)$.

Example: If $B = \left(\frac{a,b}{\mathbb{R}}\right)$, then $B \cong M_2(\mathbb{R})$ or $B \cong \mathbb{H}$ where $\mathbb{H}$ are the Hamiltonian quaternions.

Example: Let F be a number field. Then $B = M_2(F) = \left(\frac{1,b}{F}\right)$ is a quatenrion algebra over F.

In a given quaternion algebra over a number field, it would be nice to have an analog of a ring of integers. This analog is a maximal order.

For the following, let R be a noetherian integral domain with field of fractions F, though late we will be interested specifically in number fields F with ring of integers R. First recall that if V is a finite-dimensional F vector space, an R-lattice of V is a finitely generated R-submodule $I \subset V$ such that $IF = V$.

Definition 1. *An R-order $\mathcal{O} \subset B$ is an R-lattice of B which is also a subring of B.*

An order is *maximal* if it is not properly contained in another R-order. Another useful type of order is an Eichler order. Later we will define a class set on Eichler orders which is used when computing Hilbert Modular forms. An *Eichler order* is an order which can be written as the intersection of two maximal orders. Additionally, two orders $\mathcal{O}$ and $\mathcal{O}'$ are said to be of the same *type or isomorphic or conjugate* if there exists $x \in B^\times$ such that $\mathcal{O}' = x\mathcal{O}x^{-1}$.

Example: Let $F = \mathbf{Q}(\sqrt{5})$. Then $R = \mathbf{Z}[\gamma]$ where $\gamma = \frac{1+\sqrt{5}}{2}$. We can take $B = \left(\frac{-1,-1}{F}\right)$ so that B has basis $1, i, j, ij$ over F and $i^2 = -1 = j^2$. Then $\mathcal{O} = R \oplus Ri \oplus Rj \oplus Rij$ is an R-order. $\mathcal{O}$ is not a maximal order. Let

$$e_1 = \frac{1}{2}(1 - \overline{\gamma}i + \gamma j)$$

$$e_2 = \frac{1}{2}(-\overline{\gamma}i + j + \gamma k)$$

$$e_3 = \frac{1}{2}(\gamma i - \overline{\gamma}j + k)$$

$$e_4 = \frac{1}{2}(i + \gamma j - \overline{\gamma} k)$$

then $\mathcal{O}_B = Re_1 \oplus Re_2 \oplus Re_3 \oplus Re_4$ is a maximal R-order of B. Conjugating by $e = \frac{1}{2}(1 - \overline{\gamma} i + \overline{\gamma} j)$ gives another maximal order $\mathcal{O}' = R \oplus Ri \oplus R((\gamma + 1) + (\gamma + 2)i + j) \oplus R((\gamma - 8) + (\gamma + 21)i - 4j + k)$. The intersection $\mathcal{O}_B \cap \mathcal{O}' = \mathcal{O} = R \oplus Ri \oplus R(\gamma + 1) + (\gamma + 2)i + j \oplus R(\gamma + (\gamma - 5)i + k)$ is an Eichler order.

Example: For a more general example of a maximal order, let F be any number field with ring of integers R. Then for any integer n, $\mathcal{O} = M_n(R)$ is an R-order of $B = M_n(F)$. If we conjugate $\mathcal{O}$ by any invertible matrix to get another order $\mathcal{O}'$, then $\mathcal{O}'$ is another maximal order of B which in general is not equal to $\mathcal{O}$. Further, if R is not a PID then there may be maximal orders $\mathcal{O}'$ which are not conjugate to $\mathcal{O}$. However, if R is a PID then every maximal R-order in B is conjugate to $\mathcal{O}$.

For the rest of this paper, F is a number field with ring of integers R. Then R is the integral closure of $\mathbf{Z}$ in F. It is useful to define integrality for quaternion algebras too.

Let $B = \left(\frac{a,b}{F}\right)$ be quaternion algebra over F. Then if $x \in B$, $x = u + vi + wj + zij$ for some $u, v, w, z \in F$. The map

$$x = u + vi + wj + zij \mapsto \overline{x} = u - vi - wj - zij$$

is called conjugation.

We are familiar with the norm and traces maps $F \to \mathbf{Q}$. With the conjugation we can define similar maps $B \to F$. They are called the reduced norm and the reduced trace and are denoted by nrd and trd respectively.

$$\mathrm{trd} : B \to F$$

$$x \mapsto x + \overline{x}$$

and

$$\mathrm{nrd} : B \to F$$

$$x \mapsto x\overline{x}.$$

Notice that for every element $x \in B$, x and $\overline{x}$ are the roots of the polynomial $t^2 - \mathrm{trd}(x)t + \mathrm{nrd}(x) \in F[t]$. We say that the element $x \in B$ is integral over R if x satisfies a monic polynomial with coefficients in R. Thus if $x \in B$, x is integral if and only if $\mathrm{nrd}(x)$ and $\mathrm{trd}(x)$ are in R. (Notice, this generalizes. If R is any integrally closed ring with field of fractions F we still get the previous statement.)

Lemma 1. *For $x \in B$, the following are equivalent:*

1. *x is integral over R;*
2. *$R[x]$ is a finitely generated R-module;*
3. *x is contained in a subring A which is a finitely generated R-module*

Now with the concept of integrality, we can see where viewing maximal orders of quaternion algebras over number fields as the analog of a the ring of integers fails. First, the ring of integers of a number field is the set of all integral elements and as we proved in class, a ring. The set of all R-integral elements of a quaternion algebra is in general not a ring or integrally closed.

Example: Take $B = M_2(\mathbb{Q})$ with elements

$$x = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}.$$

Then $x^2 = y^2 = 0$, so x and y are integral over $\mathbb{Z}$ but not in $M_2(\mathbb{Z})$. Additionally $\mathrm{nrd}(x + y) = 1/4$ so $x + y$ is not integral. Thus the set of all integral elements is not closed under addition and can't be a ring.

Even though maximal orders are not integrally closed or even unique, orders in general still have a tie to integrality.

Lemma 2. *Let B be a quaternion algebra over the number field F. Let $\mathcal{O} \subset B$ be a subring of B such that $\mathcal{O}F = B$. Then $\mathcal{O}$ is an R-order if and only if every $x \in \mathcal{O}$ is integral.*

2 Discriminants

With norms and traces we can define discriminants and see that as in the commutative case, these are related to ramification and maximality of orders. For a quaternion algebra B with elements x_1, x_2, x_3, x_4, we define the function

$$\mathrm{d}(x_1, x_2, x_3, x_4) = \det(\mathrm{trd}(x_i x_j))_{i,j=1,2,3,4}.$$

Let $\mathcal{O}$ be an order. Notice that if $x_1, x_2, x_3, x_4 \in \mathcal{O}$ then $\mathrm{d}(x_1, x_2, x_3, x_4) \in R$. Then we can define the discriminant of $\mathcal{O}$ to be the R-ideal

$$\mathrm{disc}(\mathcal{O}) = \{\mathrm{d}(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \mathcal{O}\}.$$

More generally, we can write $\mathcal{O}$ as,

$$\mathcal{O} = \mathfrak{a}_1 x_1 \oplus \mathfrak{a}_2 x_2 \oplus \mathfrak{a}_3 x_3 \oplus \mathfrak{a}_4 x_4$$

where $\mathfrak{a}_i$ are fractional ideals of R and x_1, x_2, x_3, x_4 is a psuedobasis for $\mathcal{O}$. From this form we see that $\mathrm{disc}(\mathcal{O}) = (\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4)^2 \mathrm{d}(x_1, x_2, x_3, x_4)$.

Example: $\mathcal{O} = R \oplus Ri \oplus Rj \oplus Rij$ be an order in $B = \left(\frac{a,b}{F}\right)$ with $a, b \in R$. Then $\mathrm{disc}(\mathcal{O})$ is generated by

$$\mathrm{d}(1, i, j, ij) = \det \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2a & 0 & 0 \\ 0 & 0 & 2b & 0 \\ 0 & 0 & 0 & -2ab \end{pmatrix} = -(4ab)^2$$

3 Valuations

Let v be a valuation of F. Then the complete field F_v has ring of integers R_v. Additionally, let π_v be a uniformizer. Then we can define $B_v = B \otimes F_v$. Then B_v is a quaternion algebra over F_v. If $\mathcal{O} \subset B$ is an R-order of B, then $\mathcal{O}_v = \mathcal{O} \otimes R_v$ is an order.

Useful fact about local norms: If F is a number field with noncomplex valuation v, then F_v has a unique unramified quadratic extension K_v. This fact gives us the following:

Lemma 3. *Let v be a noncomplex place of F. Then there is a unique quaternion algebra B_v over F_v which is a division ring up to F_v-algebra isomorphism.*

As $\mathbb{C}$ is algebraically closed, there is no division quaternion algebra. Over $\mathbb{R}$ the unique division algebra is the Hamiltonians, $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$. Over $\mathbb{R}$, if $B = \left(\frac{a,b}{\mathbb{R}}\right)$ is not a division algebra, then $B \cong M_2(\mathbb{R})$.

If v is nonarchimedean, then F_v has K_v as its unique unramified extension. Thus to create a division ring over F_v, $B_v \cong \left(\frac{K_v, \pi_v}{F_v}\right)$. Similarly, if B_v is not a division ring, then $B_v \cong M_2(F_v)$.

Definition 2. *If B_v is a division ring, we say B_v is ramified or that B is ramified at v. Otherwise we say B_v is split or B splits at v.*

Let S be the set of ramified places of B. Then S is finite and even and characterizes B up to isomorphism. Further, as S is finite we have the following definition:

Definition 3. *The discriminant of B,*

$$\mathcal{D}(B) = \prod_{v \in S, v \text{ finite}} v,$$

i.e., $\mathcal{D}$ is the product of the ramified primes.

Studying Eichler orders and Maximal orders locally return global properties. If B_v is ramified, B_v has a unique maximal order. However, in the split case we have the following:

Lemma 4. *Let $\mathcal{O}$ be an order of $M_2(K_v)$. TFAE:*

1. *$\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$ for two unique maximal orders $\mathcal{O}_1$ and $\mathcal{O}_2$.*
2. *For some $n \in \mathbf{Z}_{>0}$, $\mathcal{O}$ is conjugate to*

$$\mathcal{O}_n := \begin{pmatrix} R & R \\ \pi^n R & R \end{pmatrix}.$$

We say such order $\mathcal{O}$ is an Eichler order of local level (π^n). Further, $\operatorname{disc}(\mathcal{O}) = (\pi^n)$. We can view a maximal order as an Eichler order of level 0. Notice that in the ramified case, B_v has a unique maximal order, which is then the unique Eichler order.

Lemma 5. *Let $\mathcal{O}$ be an order of B. Then $\mathcal{O}$ is maximal if and only if $\mathcal{O}_v$ is maximal for every finite place v. Further, $\mathcal{O}$ is maximal if and only if* $\operatorname{disc} \mathcal{O} = \mathcal{D}$.

Using this lemma one can show:

Lemma 6. *Let $\mathcal{O}$ be an order of B. Then $\mathcal{O}$ is an Eichler order if and only if $\mathcal{O}_v$ is an Eichler order for every finite place v.*

Now we can define the level of an Eichler order.

Definition 4. *Let $\mathcal{O}$ be an Eichler order of B. Then the level $\mathcal{N}$ of $\mathcal{O}$ is the unique integral ideal $\mathcal{N}$ in R such that $\mathcal{N}_v$ is the level of each $\mathcal{O}_v$ at each finite place v of F. I.e., $\mathcal{N}$ is the product of the local levels.*

4 Class numbers and Class not-groups

For this section we fix B to be the quaternion algebra $B = \left(\frac{a,b}{F}\right)$ over the number field F with ring of integers R. We define a *right ideal*, I, to be a right R-lattice in B such that $IF = B$. [1] Left and two-sided ideals are defined similarly. This definition gives us the analogue of a fractional ideal in the commutative case. Notice that if I is additionally a subring of B, then it is an order. [1] We say two ideals I and J are in the same *right ideal class* if $I = xJ$ for some $x \in B$.

Above we gave a very general definition of an ideal, it did not require giving which order with which the ideal is associated. Given an ideal I, we can build orders of B called the left and right orders of I:

$$\mathcal{O}_l(I) = \{x \in B : xI \subset I\}$$

$$\mathcal{O}_r(I) = \{x \in B : Ix \subset I\}.$$

Further, I is a two-sided ideal iff $\mathcal{O}_l(I) = \mathcal{O}_r(I)$.

We define an ideal I to be principal if there exists some $x \in B$ such that $I = \mathcal{O}_l(I)x = x\mathcal{O}_r(I)$ and the inverse of I to be $I^{-1} = \{x \in B : IxI \subset I\}$. If I is principal or if I is an ideal of an Eichler order then

$$II^{-1} = \mathcal{O}_l(I), I^{-1}I = \mathcal{O}_r(I).$$

In general we can only say

$$II^{-1} \subset \mathcal{O}_l(I), I^{-1}I \subset \mathcal{O}_r(I).$$

In general, if I is a right fractional ideal of the order $\mathcal{O}$, we say I is invertible if there exists a left fractional $\mathcal{O}$-ideal I^{-1} such that $I^{-1}I = \mathcal{O}$. If I is invertible, $I^{-1} = \{x \in B : xI \subset \mathcal{O}\}$.

Now we have the machinery to talk about ideal classes. With these definitions, we have the possibility for right, left, or two-sided class sets. We define a right class set, denoted $\mathrm{Cl}(\mathcal{O})$, of an order $\mathcal{O} \subset B$ to be the right ideal classes of invertible right fractional ideals. The left and two-sided ideal class sets are definite similarly. While the right and left ideal classes do not form groups, the two-sided ideal classes do.

If two orders $\mathcal{O}$ and $\mathcal{O}'$ are such that the there exists an ideal I with $\mathcal{O}_l(I) = \mathcal{O}$ and $\mathcal{O}_r(I) = \mathcal{O}'$ then we say $\mathcal{O}$ and $\mathcal{O}'$ are *linked*. This is an equivalence relation on orders called *linkage classes*.

Example: Let $\mathcal{O}$ and $\mathcal{O}'$ be any two maximal orders of B. Then $I = \mathcal{O}\mathcal{O}'$ is an ideal and $\mathcal{O} \subset \mathcal{O}_l(I)$ and $\mathcal{O}' \subset \mathcal{O}_r(I)$. However, $\mathcal{O}$ and $\mathcal{O}'$ are maximal so we must have equality, thus $\mathcal{O}$ and $\mathcal{O}'$ are linked.

Lemma 7. *Linked orders have the same number of (left or right) ideal classes.*

Corollary 1. *All maximal orders have the same number of (left or right) ideal classes.*

With this corollary we can define the class number of B with respect to R. This is the order of $\mathrm{Cl}(\mathcal{O})$ where $\mathcal{O}$ is any maximal order. Further, the type number of B is the number of conjugacy classes of maximal orders of B. For example, if R is a PID then $B \cong M_2(F)$ has type number 1 as all maximal orders are conjugate to $M_2(R)$.

Lemma 8. *The following are equivalent:*

1. *Two orders $\mathcal{O}$ and $\mathcal{O}'$ are of the same type.*
2. *Two orders $\mathcal{O}$ and $\mathcal{O}'$ are linked by a principal ideal I.*

As with most things in number theory, to study the class number of B and the type number of B it is useful to study the local case. As before, K_v is a local field with uniformizing element π and ring of integers R_v. Then B_v is either the unique division quaternion algebra or $B_v \cong M_2(K_v)$. First we'll examine the split case.

4.1 Split Case: $B_v \cong M_2(K_v)$

Punchline: There are infinitely many maximal orders, but they're all conjugate. Thus the type number is 1. Similarly, the class number is 1. Further, to classify conjugacy classes of Eichler orders, the only invariant necessary is the level, which can be any positive integer. Thus the class sets are the same for Eichler orders of the same level.

4.2 Ramified Case: B_v a division algebra

I.e., The awesome case.

Lemma 9. *The set of integral elements of B_v forms the unique maximal order $\mathcal{O}_v$.*

Further, the discriminant of $\mathcal{O}_v$ is (π). Thus the class number is 1, the type number is 1, and as there is only one maximal order, there is only one Eichler order, the maximal order itself.

5 Adeles/Ideles

In class we examined the adeles and ideles of a number field F, $\mathbb{A}_F$ and $\mathbb{I}_F$. In the same way we defined B_v by tensoring, we can create the adeles of a quaternion algebra.

As usual

$$\mathbb{A}_F = \prod_v{}' F_v = \{(x_v)_v \in \prod_v F_v : |x_v|_v \leq 1 \text{ for all but finitely many places } v\}$$

and

$$\mathbb{I}_F = \mathbb{A}_F^\times = \{(x_v)_v \in \prod_v F_v^\times : x_v \in R_v^\times \text{ for all but finitely many places } v\}.$$

The adele ring of B is defined to be $\mathbb{A}_B = B \otimes_F \mathbb{A}_F$. As $\mathbb{A}_F^\times = \mathbb{I}_F$, we define $\mathbb{I}_B = \mathbb{A}_B^\times$. Further, the topology on $\mathbb{A}_B$ is the restricted product topology (just like with $\mathbb{A}_F$) and as with the ideles, the topology on $\mathbb{I}_B$ is the induced topology on the product topology with respect to the embedding of $\mathbb{I}_B$ into $\mathbb{A}_B \times \mathbb{A}_B$ sending x to (x, x^{-1}).

For an order $\mathcal{O} \subset B$, we've defined the set $\mathrm{Cl}(\mathcal{O})$ to be isomorphism classes of right invertible fractional $\mathcal{O}$-ideals. Then, as with the number field case, we have the following theorem:

Theorem 1 (4)**.** *The set* $\mathrm{Cl}(\mathcal{O})$ *is in bijection with* $B^\times \setminus \hat{B}^\times/\hat{O}^\times$

Proof: Take I to be an invertible right fractional $\mathcal{O}$-ideal. Then I is locally principal, so $I_v = x_v\mathcal{O}_v$ is principal for all v primes of R. Thus we associate I with $(x_v\mathcal{O}_v)_v = \hat{x}\hat{O} \subset \hat{B}$. This association is unique up to units of $\mathcal{O}_v$, i.e., the associated $\hat{x}$ is unique in $\hat{B}^\times/\hat{O}^\times$.
Going the other direction, with a given $\hat{x} \in \hat{B}^\times/\hat{O}^\times$, just take the unique ideal I defined by $I = \hat{x}\hat{O} \cap B$.

We could have also chosen to work on the left instead of the right. The bijection between the two $\mathrm{Cl}(\mathcal{O})$-set is then given by the bijection betwen $\hat{B}^\times/\hat{O}^\times \to \hat{O}^\times \setminus \hat{B}^\times$, $\hat{x} \mapsto \overline{\hat{x}}$.

Again, we could have chosen to work with two-sided ideals. If we take $N(\hat{\mathcal{O}}) = \{\hat{x} \in \hat{B}^\times : \hat{x}\hat{\mathcal{O}} = \hat{\mathcal{O}}\hat{x}\}$, then the two-sided class group of $\mathcal{O}$ is in bijection with $N(\mathcal{O}) \setminus N(\hat{O})/\hat{O}^\times$.

6 Application: Hilbert Modular Forms

For this section, we restrict F to being a totally real number field with $[F : \mathbf{Q}] = n$ and real places $v_1, ..., v_n$. Let $\mathcal{H}$ be the complex upper half plane. Then the group $\mathrm{Gl}_2^+(F)$ acts on $\mathcal{H}^n$ by coordinate wise linear fractional transformations. For $\gamma \in \mathrm{Gl}_2^+(F)$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

γ acts on $z \in \mathcal{H}^n$ by

$$\gamma z = (\gamma_i z_i) = \left(\frac{a_i z_i + b_i}{c_i z_i + d_i} \right)_{i=1,\ldots,n},$$

where $\gamma_i = v_i(\gamma)$. Let $\mathcal{N}$ be a non-zero ideal of R, the ring of integers of F. Define

$$\Gamma_0(\mathcal{N}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2^+(R) : c \in \mathcal{N} \right\}.$$

Definition 5. *A Hilbert modular form of parallel weight 2 and level $\mathcal{N}$ is a holomorphic function $f : \mathcal{H}^n \to \mathbb{C}$ such that*

$$f(\gamma z) = f\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \cdots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right) = \left(\prod_{i=1}^{n} \frac{(c_i z_i + d_i)^2}{\det \gamma_i} \right) f(z)$$

for all $\gamma \in \Gamma_0(\mathcal{N})$.

The space of parallel weight 2 and level $\mathcal{N}$ modular forms is a finite dimensional $\mathbb{C}$-vector space and is denoted by $M_2(\mathcal{N})$. In the Hilbert modular form case, the cusps are $(z_i) \in \overline{\mathbb{R}}^n$ where $z_i = v_i(z)$ for some $z \in F \cup \{\infty\}$. A form f is a cusp form if f vanishes at the cusps. If $[F : \mathbf{Q}] = n > 1$, we do not need the condition that the form must be holomorphic at the cusps as this follows from Koecher's principal.

Let B be a quaternion algebra over F. We can write the real places of F so that $v_1, \cdots, v_r$ are the places where B is split and $v_{r+1}, \cdots, v_n$ are the places where B is ramified. Then

$$B \oplus_{\mathbf{Q}} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{n-r}.$$

If $r = 0$, so that B is ramified at all the real places, then we say that B is definite. Otherwise we say B is indefinite.

If B is indefinite and $n > 1$, then we have a map

$$\iota_\infty : B \hookrightarrow M_2(\mathbb{R})^r$$

which corresponds to the embeddings $v_1, \cdots, v_r$. Then $B_+^\times$ embeds into $\mathrm{Gl}_2^+(\mathbb{R})^r$ where

$$B_+^\times = \{\gamma \in B^\times : \det\gamma_i = (\mathrm{nrd}\gamma)_i > 0 \text{ for } i = 1, ...r\}$$

and so acts on $\mathcal{H}^r$ coordinate wise. Let $\mathcal{O}_B$ be a maximal order of B and $\mathcal{O} \subset \mathcal{O}_B$ be an order of level $\mathcal{N}$. Let $\mathcal{O}_+^\times = \mathcal{O}^\times \cap B_+^\times$. If we further restrict F to having narrow class number one, then $\mathcal{O}_+^\times = R^\times \mathcal{O}_1^\times$ where $\mathcal{O}_1^\times$ are the elements of $\mathcal{O}$ with reduced norm 1. Then we can define $\Gamma = \Gamma_0^B(\mathcal{N}) = \iota_\infty(\mathcal{O}_+^\times) \subset \mathrm{GL}_2^+(\mathbb{R})^r$.

Definition 6. *Let B be indefinite. A quaternionic modular form of parallel weight 2 and level $\mathcal{N}$ is a holomorphic function $f : \mathcal{H}^n \to \mathbb{C}$ such that*

$$f(\gamma z) = f\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \cdots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right) = \left(\prod_{i=1}^n \frac{(c_i z_i + d_i)^2}{\det \gamma_i}\right) f(z)$$

for all $\gamma \in \Gamma_0^B(\mathcal{N})$.

The space of parallel weight 2 and level $\mathcal{N}$ quaternionic modular forms is a finite dimensional $\mathbb{C}$-vector space and is denoted by $M_2^B(\mathcal{N})$. If B is a division ring (so we are working in the case $0 < r < n$) then there are no cusps, and so $M_2^B(\mathcal{N}) = S_2^B(\mathcal{N})$, the quaternionic cusp forms.

Now let's work out the definite case, so $r = 0$. Again $\mathcal{O} \subset \mathcal{O}_B$ is an Eichler order of level $\mathcal{N}$. Recall that the set of invertible right $\mathcal{O}$-ideal classes $\mathrm{Cl}(\mathcal{O})$ is finite and its order is independent of the choice of Eichler order of level $\mathcal{N}$.

Definition 7. *Let B be definite and $\mathcal{O}$ be an Eichler order of level $\mathcal{N}$. Then a quaternion modular form for B of parallel weight 2 and level $\mathcal{N}$ is a map*

$$f : Cl(\mathcal{O}) \to \mathbb{C}.$$

Then $M_2^B(\mathcal{N})$ is a $\mathbb{C}$-vector space of dimension equal to the order of the class set. Here we define cusp forms to be modular forms of B which are orthogonal with respect to the Petersson inner product (see van der Geer, *Hilbert Modular Surfaces* for the definition) to the subspace of constant functions. As before they are denoted by $S_2^B(\mathcal{N})$.

Theorem 2. *(Eichler-Shimizu-Jacquet-Langlands). [3] Let B be a quaternion algebra over F of discriminant $\mathcal{D}$. Let $\mathcal{N}$ be an ideal coprime to $\mathcal{D}$. Then there is an injective map of Hecke modules*

$$S_2^B(\mathcal{N}) \hookrightarrow S_2(\mathcal{D}\mathcal{N})$$

with image consisting of the Hilbert cusp forms which are new at all primes dividing $\mathcal{D}$.

(I haven't given the definition of new, it can be found in [3]). If we take B to be a quaternion algebra of discriminant $\mathcal{D} = (1)$, then

$$S_2^B(\mathcal{N}) \cong S_2(\mathcal{N}).$$

Such quaternion algebra B with $\mathcal{D} = 1$ is one such that it is ramified only at real places (as we are taking F to be totally real). Recall that a quaternion algebra must be ramified at an even number of places. So when $n = [F : \mathbf{Q}]$ is even, we just take B to be the definite quaternion algebra ramified at all the real places. When n is odd, we take B to be the indefinite quaternion algebra ramified at all but one real place (and again unramified at all finite places).

This correspondence is what Voight and Dembele exploit to compute the space $S_2(\mathcal{N})$ of Hilbert cusp forms of level $\mathcal{N}$ over F (a totally real number field with narrow class number 1) as a Hecke module.

7 References

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