Math 581b, Fall 2010, Homework 6

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Due: Wednesday, November 10, 2010

Do the following problems, and turn them in by the beginning of class on Wednesday, November 10, 2010. There are 3 problems. NOTE: In case I haven't mentioned it before, you *are* encouraged to talk to other students in class about homework (if you get a lot of help on a problem from somebody, acknowledge them).

1. Read some background related to your final project (e.g., papers, chapters of a book, whatever). Briefly summarize what you read here.

2. (This problem is based on a problem in Marcus, *Number Fields*, page 156.) Let $K \subset \mathbf{C}$ be a number field that is a Galois extension of $\mathbf{Q}$.

   (a) Prove that K has degree 1 or 2 over $K \cap \mathbf{R}$. (Hint: write down a quadratic polynomial with real coefficients satisfied by a generator of K.)

   (b) Prove that $K \cap \mathbf{R}$ is a Galois extension of $\mathbf{Q}$ if and only if $K \cap \mathbf{R}$ has no non-real embeddings into $\mathbf{C}$.

   (c) Prove that $K \cap \mathbf{R}$ is a Galois extension of $\mathbf{Q}$ if and only if the subgroup of $\mathrm{Gal}(K/\mathbf{Q})$ generated by complex conjugation (acting on K) is a normal subgroup. (Obviously, you can state any theorems from Galois theory.)

   (d) Let $U = \mathcal{O}_K^*$ be the group of units of the ring of integers of the number field K. Prove that $U/(U \cap \mathbf{R})$ is finite if and only if complex conjugation is in the center of the group $\mathrm{Gal}(K/\mathbf{Q})$. (Hint: compute "$r + s - 1$" for both K and $K \cap \mathbf{R}$, in various cases.)

3. (This problem is based on a problem in Marcus, *Number Fields*, page 156. It also uses results from the previous problem above.)

   (a) Let u be a unit in $\overline{\mathbf{Z}}$, which we view as a subring of the complex numbers $\mathbf{C}$. Prove that the complex conjugate $\overline{u}$ and the absolute value $|u| = \sqrt{u\overline{u}}$ are both also units in $\overline{\mathbf{Z}}$.

   (b) Let $K \subset \mathbf{C}$ be the Galois closure of a cubic field $M = \mathbf{Q}(\sqrt[3]{p})$, where p is a prime number, so $[K : \mathbf{Q}] = 6$ and $\mathrm{Gal}(K/\mathbf{Q}) = S_3$, the symmetric group on 3 symbols. Prove that $U/(U \cap \mathbf{R})$ is infinite, where $U = \mathcal{O}_K^*$.

   (c) Show that the field K in part 3b above contains a unit having absolute value 1 but which is not a root of unity. (Hint: Let $u \in U$ be a unit from part 3b with the property that $u^n \notin \mathbf{R}$ for any integer $n \neq 0$, and consider $u/\overline{u}$.)

   (d) Use Sage (or Magma or Pari or Mathematica...) to find an explicit example that illustrates part 3c.