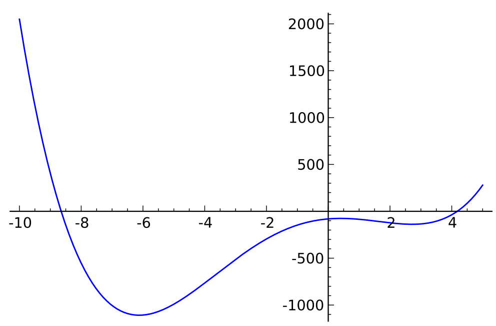
Math 581b, Fall 2010, Homework 5

Due: Friday, November 5, 2010

Do the following problems, and turn them in by the beginning of class on Friday, November 5, 2010. (I'm giving you an extra two days, because I assigned this homework set late.) There are 3 problems.

1. The purpose of this problem is to get you to start thinking about your final project. Choose a *rough topic* for your final project, and describe it in less than a page. You may find the syllabus from the Math 581b website useful, since it says that one of the goals of the course is to learn about "number fields, (degree 1) function fields, adeles and ideles, Galois cohomology groups, local fields, class fields and the Artin reciprocity map, elliptic curves." That list may give you some ideas for a project topic. You can also see 14 fine student projects listed here: `http://wstein.org/129-05/final_papers/` (amazingly, these were all projects by *undergraduates*). There are also many interesting projects by UW students from 2007 here: `http://wiki.wstein.org/ant07/projects`.

2. Let α be any root of the polynomial $f(x) = x^4 + 4x^3 - 36x^2 + 27x - 81$, and let $K = \mathbf{Q}(\alpha)$. Find (by any means at all, e.g., Pari, Magma, Sage) generators for the group of units U_K of the ring of integers $\mathcal{O}_K$ of K. This is a plot of $f(x)$:



3. Let U be the group of units $x + y\sqrt{5}$ of the ring of integers of $K = \mathbf{Q}(\sqrt{5})$.

   (a) Prove that the set S of units $x + y\sqrt{5} \in U$ with $x, y \in \mathbf{Z}$ is a subgroup of U. (The main point is to show that the inverse of a unit with $x, y \in \mathbf{Z}$ again has coefficients in $\mathbf{Z}$.)

   (b) Let U^3 denote the subgroup of cubes of elements of U. Prove that $S = U^3$ by showing that $U^3 \subset S \subsetneq U$ and that there are no groups H with $U^3 \subsetneq H \subsetneq U$.