Math 581b, Fall 2010, Homework 2

Assigned: Wednesday, October 6, 2010

Do the following problems, and turn them in by the beginning of class on Wednesday, October 13, 2010. There are 2 problems.

1. For each of the following rings, determine whether or not it is a Dedekind domain. (Obviously, explain your reasoning.)

   (a) The polynomial ring $k[x,y]$ in two variables, where k is a field.

   (b) The ring $\mathbf{Z}_p$ of p-adic integers.

   (c) The noncommutative (!) ring $M_2(\mathbf{Z})$ of 2×2 integer matrices.

   (d) The ring $\overline{\mathbf{Z}}$ of all algebraic integers.

   (e) The field $\overline{\mathbf{Q}}$ of all algebraic numbers.

   (f) The ring $\mathbf{Z}[\sqrt{3}]$.

   (g) The ring $\mathbf{Z}[\sqrt{5}]$.

   (h) The affine coordinate ring $\mathbf{C}[x,y]/(y^2 - x^3)$ of a cuspidal cubic over the complex numbers.

   (i) The ring $\mathbf{Z}[\frac{1}{2}]$ of rational numbers whose denominator is a power of 2.

   (j) The ring $\mathbf{Z}[\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \ldots]$ generated by the square roots of all prime numbers.

   (k) The quotient ring $\mathbf{Z}[x]/(x^2)$, which is a ring with an element whose square is 0.

   (l) The quotient ring $\mathbf{Z}[x]/(x^2 - 1)$.

   (m) The direct sum $\mathbf{Z} \oplus \mathbf{Z}$, so the direct sum of two copies of $\mathbf{Z}$.

   (n) Any finite integral domain.

2. Let K be a quadratic field (i.e., a number field of degree 2) and let $\alpha \in K$. If $\mathrm{Norm}_{K/\mathbf{Q}}(\alpha) \in \mathbf{Z}$ and $\mathrm{Tr}_{K/\mathbf{Q}}(\alpha) \in \mathbf{Z}$, does it necessarily follow that $\alpha \in \mathcal{O}_K$? What if K has degree 3 instead of 2?