Math 581b, Fall 2010, Homework 1

September 29, 2010

Do the following problems, and turn them in by the beginning of class on Wednesday, October 6, 2010. There are 5 problems.

1. Prove *as directly as you can* that if $K = \mathbf{Q}(\sqrt{-1})$, then $\mathcal{O}_K = \mathbf{Z}[\sqrt{-1}]$.
2. (a) Compute—in any way—the minimal polynomial of the algebraic number $\sqrt{2}+\sqrt{3}+\sqrt{5}$. Using a computer is allowed (show how), though not essential.

   (b) The minimal polynomial of $\sqrt{p_1}+\sqrt{p_2}+\sqrt{p_3}+\cdots+\sqrt{p_n}$, where p_i is the ith prime, is called the nth Swinnerton-Dyer polynomial. Is every Swinnerton-Dyer polynomial a monic polynomial in $\mathbf{Z}[x]$?
3. Suppose $\mathcal{O}_K$ is the ring of integers of a number field K and $\beta \in \overline{\mathbf{Q}}$ is a root of a monic polynomial $f(x) \in \mathcal{O}_K[x]$. Prove that β is an algebraic integer, i.e., β is a root of some monic integral polynomial $g(x) \in \mathbf{Z}[x]$.
4. Is the following number an algebraic integer (i.e., the root of a monic integral polynomial in $\mathbf{Z}[x]$)?

$$\sqrt[20001]{\sqrt[5]{7}+8} \;+\; \sqrt[2010]{\sqrt[7]{3}+\sqrt{2}} \;+\; 1$$

5. (a) Is every prime ideal of the ring $\mathbf{Z}[X]$ of polynomials over $\mathbf{Z}$ maximal?

   (b) Prove that in the following four rings every nonzero prime ideal is maximal (hence each ring has "Krull dimension 1"):

$$\mathbf{Z}, \quad \mathbf{Q}[X], \quad \mathbf{Z}[\sqrt{-1}], \quad \mathbf{Q}[X,Y]/(Y^2 - X^3 - 1).$$