Bernoulli Numbers and Various Consequences

Jordan Nakamura

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Introduction

Jacob Bernoulli was a mathematician who created a class of numbers called, not surprisingly, the Bernoulli Numbers. Bernoulli was studying formulas concerning summing the k^{th} powers of n integers, i.e [2]

$$1+2+3+4+...(n-1)=\frac{n(n-1)}{2}$$
$$1^2+2^2+...(n-1)^2=\frac{n(n-1)(2n-1)}{6}$$
$$\vdots \qquad \qquad \vdots$$
$$1^k+2^k+3^k...+(n-1)^k=S_k(n)$$

It was through this research on determining the coefficients of these polynomials that he created the Bernoulli numbers.

Bernoulli Numbers

The Bernoulli numbers are a sequence of rational numbers discovered by Jacob Bernoulli, and these numbers are used in many different areas of mathematics. Some that this paper addresses are the Riemann-zeta function and Kummer Congruencies. The Bernoulli numbers can be defined directly or recursively. The recursive definition is

$$(m+1)B_m=-\sum_{k=0}^{m-1}\binom{m+1}{k}B_k$$

where $B_0=1$.[1]

The direct definition refers to the coefficients of the power series of

$$\frac{t}{e^t-1}=\sum_{m=0}^{\infty}B_m(\frac{t^m}{m!})$$

This fact can be shown by the following: [2]

$$\frac{t}{e^t-1}=\sum_{m=0}^{\infty}b_m(\frac{t^m}{m!}) \ \ \forall m \ s.t \ b_m=B_m$$

$$\Rightarrow \quad t = (e^t - 1) \cdot \sum_{m=0}^{\infty} b_m (\frac{t^m}{m!})$$

$$\Rightarrow \; t = (\sum_{n=0}^{\infty} \frac{t^n}{n!} - 1) \cdot \sum_{m=0}^{\infty} b_m (\frac{t^m}{m!})$$ *by the power series definition of* e^t

$$\Rightarrow t = (\sum_{n=1}^{\infty} \frac{t^n}{n!}) \cdot \sum_{m=0}^{\infty} b_m (\frac{t^m}{m!})$$

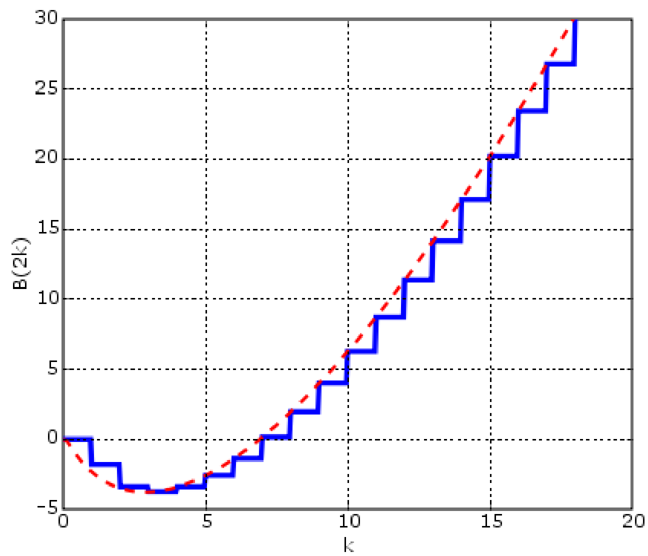
$$\Rightarrow t = b_0 t + [b_1 t^2 + \frac{b_0}{2!} t^2] + [\frac{b_2}{2!} t^3 + \frac{b_1}{2!} t^3 + \frac{b_0}{3!}] + ...$$

$\Rightarrow$ *If you look at the coefficients of values of* t^{m+1} *(especially when m = 0)*
You get that $b_0 = 1$, *and when you look at multiple values of m, you get:*
$\sum_{k=0}^{m} \binom{m+1}{k} b_k = 0$ *which is the definition of a Bernoulli number. Since* $b_0 = B_0$, *it follows that* $b_m = B_m$.

Now that the definition of Bernoulli numbers has been explained, here are the first several numbers:

$B_0 = 1$	$B_4 = -\frac{1}{30}$	$B_8 = -\frac{1}{30}$	$B_{12} = \frac{-691}{2730}$
$B_1 = -\frac{1}{2}$	$B_5 = 0$	$B_9 = 0$	$B_{13} = 0$
$B_2 = \frac{1}{6}$	$B_6 = \frac{1}{42}$	$B_{10} = \frac{5}{66}$	$B_{14} = \frac{7}{6}$
$B_3 = 0$	$B_7 = 0$	$B_{11} = 0$	$B_{15} = 0$

One will note that for all odd indices other than one, the Bernoulli number evaluates to 0, and the even indices always change sign. Another observation is that $|B_{2m}|$ keeps decreasing until B_6 at which point it begins to increase. In fact, the graph of what the Bernoulli numbers look like is as follows:



The red dotted line is represented by the equation $\log(4\sqrt{\pi k}(\frac{k}{\pi e})^{2k})$ and the blue is $\log\left|B_{\lfloor 2k \rfloor}\right|$.
[3] In order to actually find out how the Bernoulli numbers behave, we can actually use a relationship between the Bernoulli numbers and the Riemann-zeta function as shown by Euler. The theorem that Euler proved was:

$$For\ \ m \in \mathbb{Z}^+, \qquad 2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}$$

During Euler's time, it was an ongoing problem to try and find a way to describe how $\zeta(2m)$ behaved but this theorem accurately describes its behavior. In order to prove this theorem, we use the fact that the partial fraction decomposition of $x\cot(x)$ is:

$$x\cot(x) = 1 - 2\sum_{m=1}^{\infty}\zeta(2m)\frac{x^{2m}}{\pi^{2m}}$$

Additionally, noting that:

$$\cos(x) = \frac{e^{ix}+e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$$

$$\Rightarrow \qquad \cot(x) = \frac{e^{ix}+e^{-ix}}{2}\cdot\frac{2i}{e^{ix}-e^{-ix}}$$

$$\Rightarrow \qquad \frac{i(e^{ix}+e^{-ix})}{e^{ix}-e^{-ix}}\cdot\left(\frac{e^{ix}}{e^{ix}}\right)$$

$$\Rightarrow \quad \frac{i(1+e^{2ix})}{e^{2ix}-1} \qquad \textit{Note that: } i(1+e^{2ix}) = i(e^{2ix}-1)+2i$$

$$\Rightarrow \quad \frac{i(e^{2ix}-1)+2i}{e^{2ix}-1} = \frac{2i}{e^{2ix}-1}+i$$

$$\Rightarrow \quad x\cot(x) = ix + \frac{2ix}{e^{2ix}-1}$$

However, note that the original definition of a Bernoulli Number that we gave us a relationship between $\frac{2ix}{e^{2ix}-1}$ *and an equation involving Bernoulli Numbers.*

$$\Rightarrow x\cot(x) = 1 + \sum_{n=2}^{\infty} B_n \frac{(2ix)^n}{n!}$$

If we study the two different series that we have set equal to xcot(x)

$$B_2 \frac{(2ix)^2}{2!} + \ldots + \frac{B_4 (2ix)^4}{4!} + \ldots$$

$$\frac{\zeta(2)\cdot 2}{\pi^2} x^2 + \frac{\zeta(4)\cdot 2}{\pi^4} x^4 + \ldots$$

we can see that we get an equation of the form:

$$-\frac{2}{\pi^{2m}}\zeta(2m) = (-1)^m \frac{2^{2m}}{(2m)!} B_{2m}$$

From this point, it is clear that this is equivalent to the original statement of

$$\textit{For } m \in \mathbb{Z}^+, \qquad 2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}$$

From this theorem, we can note a couple of important things about the Bernoulli numbers, one being that we have a rough estimate of how large B_{2m} might become.

$$|B_{2m}| > \frac{2(2m)!}{(2\pi)^{2m}}$$

Obviously, the Bernoulli numbers grow at a fast rate, which brings up a question about their calculations. From this formula we get:

$$|B_m| = \frac{2m!}{(2\pi)^m}\zeta(m)$$

which seems like a reasonable way to calculate Bernoulli numbers. If we code up a program in SAGE we should be able to find Bernoulli numbers now.

```
def bern(m):

    numerator = 2*factorial(m)

    denominator = (2*pi)^m

    riem = zeta(m)

    return numerator/denominator * riem
```

And using this function, we see that we get this as an output which *seems* to match SAGE's own implementation of the Bernoulli numbers.

```
for i in range(2,6,2):

    print(N(bern(i)))

    print(N(bernoulli(i)))

>>0.166666666666667
>>0.166666666666667
>> 0.0333333333333333
>> 0.0333333333333333
```

However, SAGE uses a better algorithm then the one previously defined above, as it finds the Bernoulli number as a fraction with numerator and denominator as integers. In our method, we have, at best, a fraction over some power of pi, which needless to say is not an integer. In order to calculate a Bernoulli number with integer numerators and denominators, we need to use a congruence discovered by Clausen and Von Staudt. The Clausen and Von Staudt congruence states that:

$$B_m \equiv -\sum_{\substack{p\ prime \\ p-1|m}} \frac{1}{p} (\bmod \mathbb{Z})$$

SAGE takes advantage of this, and thus is able to give a Bernoulli number in the form of a fraction of two integers.

```
bernoulli(10)

>> 5/66
```

Kummer Congruences

An interesting fact, is that mathematician Ernst Kummer had a similar congruence to that of Clausen and Von Staudt that he used to show a specific case of Fermat's Last Theorem and that also relates to ideal class theory [4]. His congruence states:

Let p be a prime and suppose that k ≥ 2 is an even integer z s.t.(p-1) does not divide z. Then the quotient $\frac{B_k}{k}$ *,as a fraction of lower terms, is such that p does not divide its denominator. Also, if h is another even integer with (p-1) not dividing k and* $k \equiv h(\text{mod}(p-1))$ *, then:*

$$\frac{B_k}{k} \equiv \frac{B_h}{h} (\text{mod } p)$$

The applications of Kummer congruencies are very vast and complicated and a couple examples of them are as follows:

- Fermat's Last Theorem is true for regular primes, as shown by Kummer himself. A Regular prime is defined as being an odd prime that does not divide the numerator of $B_2,\ B_4,\ B_6,\ ...,\ B_{p-3}$. This is also significant because the number of regular primes is thought to be infinite,
- Use in p-adic L-functions, such as use in Iwasawa theory [4].

Bernoulli Polynomials

The Bernoulli polynomials are defined as such:

$$B_m(x) = \sum_{k=0}^{m} \binom{m}{k} B_k x^{m-k}$$

Some examples of polynomials are:

$$B_1(x) = x - \frac{1}{2} \qquad B_2(x) = x^2 - x + \frac{1}{6}$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \qquad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

Earlier in the paper, we defined $S_m(n)$, and we will use this to define a theorem that will allow us to relate $S_m(n)$ and the Bernoulli polynomials and numbers: [2]

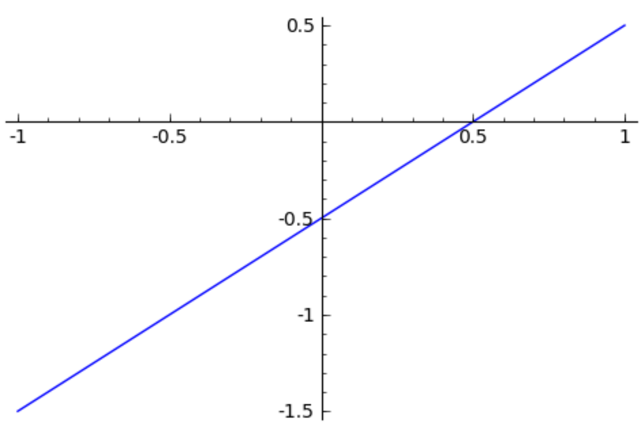
For $m \geq 1$, $S_m(n)$ *will satisfy the equation:*

$$(m+1)S_m(n) = \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}$$

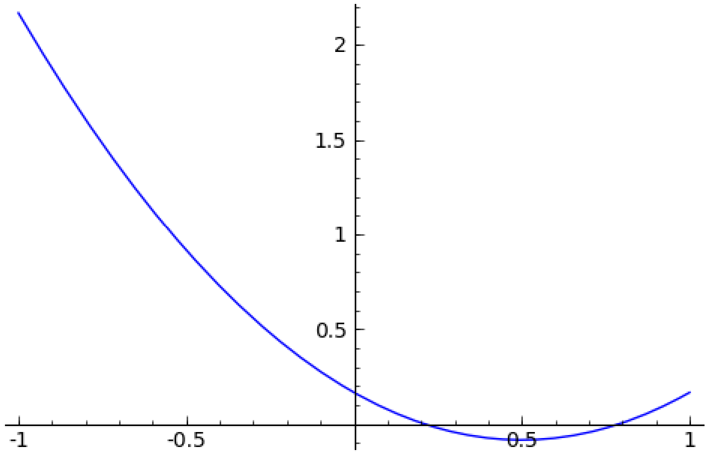
From this theorem, we can get the relationship:

$$S_m(n) = \frac{1}{m+1}(B_{m+1}(n) - B_{m+1})$$

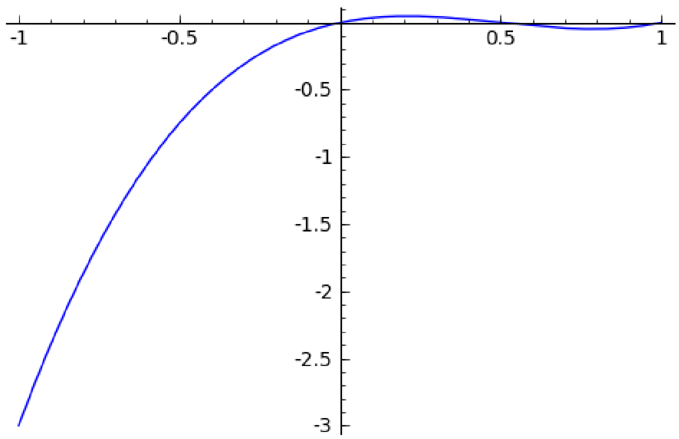
The Bernoulli polynomials are used in the study of the Riemann-zeta function, and also in the Hurwitz-zeta function. The Bernoulli polynomials have interesting properties which are not immediately obvious from looking at their graphs. If we plot some of the Bernoulli polynomials in SAGE we get these results:



For B_1



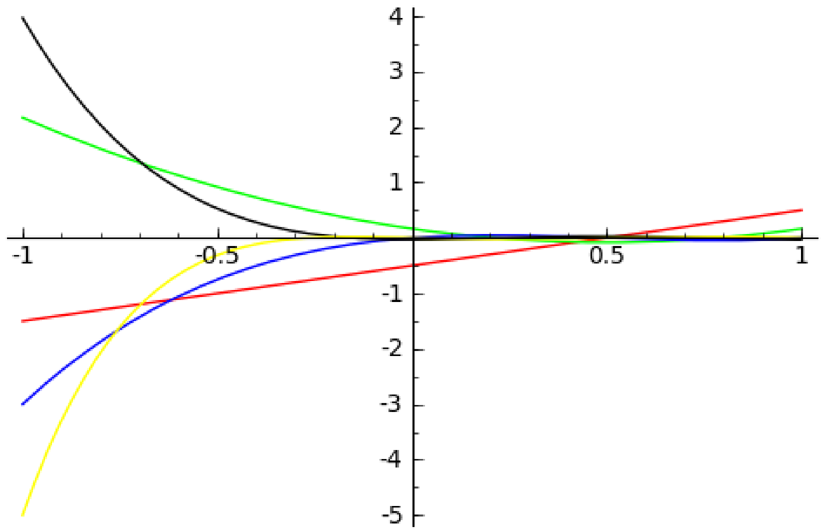
For B_2



For B_3

If we use SAGE to graph the first few on the same plot we get this result using this code:

```
a = plot(bernoulli_polynomial(x,1), rgbcolor= 'red')
a += plot(bernoulli_polynomial(x,2),  rgbcolor= 'green')
a += plot(bernoulli_polynomial(x,3),  rgbcolor = 'blue')
a += plot(bernoulli_polynomial(x,4),  rgbcolor = 'black')
a += plot(bernoulli_polynomial(x,5), rgbcolor = 'yellow')
plot(a)
```



One of the amazing things about these polynomials is that when scaled properly, they will approach the sine and cosine functions. [5] The polynomials have to be scaled because the Bernoulli Numbers increase very rapidly, and sin(x) obviously is bounded by 1 and -1.

These polynomials have been used in order to evaluate certain Dirichelet series in terms of Bernoulli polynomials. This is achieved by using the Fourier expansion of a class of certain Bernoulli polynomials. *(For more information on this topic, see Balanzario, and Sanchez-Ortiz's paper on this subject.)*

References

[1] Shanks, Daniel. *Solved and Unsolved Problems in Number Theory*. New York, N.Y.: Chelsea Pub., 1985. Print.

[2] Ireland, Kenneth F., and Michael I. Rosen. *A Classical Introduction to Modern Number Theory*. New York: Springer-Verlag, 1990. Print.

[3] "File:Bernoulli Numbers Logarithmic Growth.png -." *Wikimedia Commons*. Web. 13 Mar. 2010. <http://commons.wikimedia.org/wiki/File:Bernoulli_numbers_logarithmic_growth.png>.

[4] "Bernoulli Number -." *Wikipedia, the Free Encyclopedia*. Web. 13 Mar. 2010. <http://en.wikipedia.org/wiki/Bernoulli_numbers>.

[5] "Bernoulli Polynomials -." *Wikipedia, the Free Encyclopedia*. Web. 13 Mar. 2010. <http://en.wikipedia.org/wiki/Bernoulli_polynomials>.

[6] Balanzario, Eugenio P., and Jorge Sanchez-Ortiz. "A Generating Function for a Class of Generalized Bernoulli Polynomials." *The Ramanujan Journal* 19.1 (2009): 9-18. Print.