Math 414

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Final Project - The Pattern of Continued Fractions for $\sqrt{d}$

When I began thinking about my project, I wanted to explore the period length of $\sqrt{d}$ when $d \in \mathbb{Z}$ and d is not quadratic residue in $\mathbb{Z}$.

After calculating in Sage:

$\sqrt{2} = [1; \overline{2}]$

$\sqrt{3} = [1; \overline{1, 2}]$

$\sqrt{5} = [2; \overline{4}]$

$\sqrt{6} = [2; \overline{2, 4}]$

$\sqrt{7} = [2; \overline{1, 1, 1, 4}]$

$\sqrt{8} = [2; \overline{1, 4}]$

$\sqrt{10} = [3; \overline{6}]$

$\sqrt{11} = [3; \overline{3, 6}]$

$\sqrt{12} = [3; \overline{2, 6}]$

$\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}]$

$\sqrt{14} = [3; \overline{1, 2, 1, 6}]$

$\sqrt{15} = [3; \overline{1, 6}]$

$\vdots$

$\sqrt{17} = [4; \overline{8}]$

$\sqrt{18} = [4; \overline{4, 8}]$

$\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$

$\sqrt{20} = [4; \overline{2, 8}]$

$\sqrt{21} = [4; \overline{1, 1, 2, 1, 1, 8}]$

$\sqrt{22} = [4; \overline{1, 2, 4, 2, 1, 8}]$

$\sqrt{23} = [4; \overline{1, 3, 1, 8}]$

$\ldots$ $\sqrt{103} = [10; \overline{6, 1, 2, 1, 1, 9, 1, 1, 2, 1, 6, 20}]$

$\sqrt{76} = [8; \overline{1, 2, 1, 1, 5, 4, 5, 1, 1, 2, 1, 16}]$

$\sqrt{54} = [7; \overline{2, 1, 6, 1, 2, 14}]$

$\sqrt{201} = [14; \overline{5, 1, 1, 1, 2, 1, 8, 1, 2, 1, 1, 1, 5, 28}]$

$\vdots$

etc.

The period length ended up being less interesting than the pattern which emerged,

$$\sqrt{d} = [a_0; \overline{a_1, a_2, a_3, \ldots, a_3, a_2, a_1, 2a_0}]$$

I will prove:

For any $d \in \mathbb{Q}$ such that $\sqrt{d} \notin \mathbb{Q}$,

$$\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}]$$

[illegible]

[illegible]

[illegible]

[illegible]

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Claim: If $d \in \mathbb{Q}$ such that $\sqrt{d} \notin \mathbb{Q}$, and $d > 1$ then $\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}]$

is the simple continued fraction for $\sqrt{d}$.

Proof: $\sqrt{d} = [a_0; a_1, a_2, \ldots, \overline{a_n, \ldots, a_{n+m-1}}]$ since $\sqrt{d}$ is a quadratic irrational. $a_0 = \lfloor \sqrt{d} \rfloor$.

Let $t_k = [a_k, a_{k+1}, \ldots]$,

So $t_k = a_k + \dfrac{1}{t_{k+1}}$ and $t_0 = \sqrt{d}$.

So $\sqrt{d} = \lfloor \sqrt{d} \rfloor + \dfrac{1}{t_1}$ and $\sqrt{d} - \lfloor \sqrt{d} \rfloor = \dfrac{1}{t_1}$.

The conjugate of t_1, $\bar{t}_1 = \dfrac{-1}{\sqrt{d} + \lfloor \sqrt{d} \rfloor}$ since

$$t_1 = \frac{1}{\sqrt{d} - \lfloor \sqrt{d} \rfloor} = \frac{\sqrt{d} + \lfloor \sqrt{d} \rfloor}{d - \lfloor \sqrt{d} \rfloor^2}$$

which means
$$\bar{t}_1 = \frac{-\sqrt{d} + \lfloor \sqrt{d} \rfloor}{d - \lfloor \sqrt{d} \rfloor^2} = \frac{-(\sqrt{d} - \lfloor \sqrt{d} \rfloor)}{(\sqrt{d} + \lfloor \sqrt{d} \rfloor)(\sqrt{d} - \lfloor \sqrt{d} \rfloor)} = \frac{-1}{\sqrt{d} + \lfloor \sqrt{d} \rfloor}.$$

$\sqrt{d} + \lfloor \sqrt{d} \rfloor > 1$ since $d > 1$. So $0 < \dfrac{1}{\sqrt{d} + \lfloor \sqrt{d} \rfloor} < 1$

and therefore $0 > \dfrac{-1}{\sqrt{d} + \lfloor \sqrt{d} \rfloor} = \bar{t}_1 > -1$

Assume $-1 < \bar{t}_k < 0$

$\bar{t}_k = a_k + \dfrac{1}{\bar{t}_{k+1}}$ so $\dfrac{1}{\bar{t}_{k+1}} = -a_k + \bar{t}_k < -a_k$, by assumption. $-a_k \leq -1$ since $a_k \geq 1$.

So $\dfrac{1}{\bar{t}_{k+1}} < -1$ which means $\bar{t}_{k+1}$ is negative.

So $1 > -\bar{t}_{k+1}$ and $-1 < \bar{t}_{k+1} < 0$.

Therefore, by induction, $-1 < \bar{t}_k < 0 \ \forall k \geq 0$

Next I will show that $t_1 = [a_1, a_2, \ldots, \overline{a_n, \ldots, a_{n+m-1}}]$ is purely periodic, so $n=1$.

Assume t_1 is not purely periodic. That means $n \geq 2$ and the period length of $t_1 = m$.

Then $a_{n-1} \neq a_{n+m-1}$, otherwise the period would begin at $n-1$. So, $t_{n-1} - t_{n+m-1}$ is nonzero.

$$t_{n-1} - t_{n+m-1} = \left(a_{n-1} + \frac{1}{t_n}\right) - \left(a_{n+m-1} + \frac{1}{t_{n+m}}\right)$$

But $t_n = t_{n+m}$ since t_n is purely periodic with period length m.

So $$t_{n-1} - t_{n+m-1} = \left(a_{n-1} + \frac{1}{t_n}\right) - \left(a_{n+m-1} + \frac{1}{t_n}\right)$$

$= a_{n-1} - a_{n+m-1}$ which is an integer since a_{n-1} and a_{n+m-1} are integers.

Since t_{n-1} and $t_{n+m-1} \in (-1, 0)$, $|t_{n-1} - t_{n+m-1}| < 1$

But the only integer in the set $(-1, 1)$ is zero. Which implies $a_{n-1} = a_{n+m-1}$, a contradiction.

Therefore, t_1 is purely periodic.

So $t_1 = [\overline{a_1, a_2, \ldots, a_m}]$.

Next I will show that $-1/\bar{t}_1 = [\overline{a_m, \ldots, a_1}]$.

$$t_1 = a_1 + \frac{1}{t_2}, \ldots, t_k = a_k + \frac{1}{a_{k+2}}, \ldots, t_m = a_m + \frac{1}{t_1}$$

and

$$\bar{t}_1 = a_1 + \frac{1}{\bar{t}_2}, \ldots, \bar{t}_k = a_k + \frac{1}{\bar{t}_{k+1}}, \ldots, \bar{t}_m = a_m + \frac{1}{\bar{t}_1}$$

So, solving the second set of equations backwards gives,

$$-\frac{1}{\bar{\epsilon}_1} = a_m - \bar{\epsilon}_m, \dots, \frac{-1}{\bar{\epsilon}_{k+1}} = a_k - \bar{\epsilon}_k, \dots, \frac{-1}{\bar{\epsilon}_2} = a_1 - \bar{\epsilon}_1$$

Substituting into the first equation gives,

$$-\frac{1}{\bar{\epsilon}_1} = a_m + \frac{1}{-1/\bar{\epsilon}_m} = a_m + \frac{1}{a_{m-1} - \bar{\epsilon}_{k-1}}$$

$$= a_m + \frac{1}{a_{m-1} + \frac{1}{-1/\bar{\epsilon}_{k-1}}} = a_m + \frac{1}{a_{m-1} + \frac{1}{a_{m-2} - \bar{\epsilon}_{m-2}}}$$

$$= a_m + \cfrac{1}{a_{m-1} + \cfrac{1}{a_{m-2} - \cfrac{1}{\ddots + \cfrac{1}{a_1 - \bar{\epsilon}_1}}}}$$

Solving recursively gives,

$$-\frac{1}{\bar{\epsilon}_1} = [\overline{a_m; a_{m-1}, \dots, a_2, a_1}].$$

Now we have two equations for

$\sqrt{d} + \lfloor \sqrt{d} \rfloor = a_0 + [a_0; \overline{a_1, a_2, \dots, a_m}] = [2a_0; \overline{a_1, \dots a_m}]$

and $\sqrt{d} + \lfloor \sqrt{d} \rfloor = -1/\bar{\epsilon}_1 = [a_m; \overline{a_{m-1}, \dots, a_1, a_m}]$

Since every real number has a unique representation as a simple continued fraction,

$a_m = 2a_0$

$a_{m-1} = a_1$

$a_{m-k} = a_k$

So $\sqrt{d} + \lfloor \sqrt{d} \rfloor = [2a_0, \overline{a_1, a_2, \dots, a_2, a_1, 2a_0}]$

and $\sqrt{d} = [a_0, \overline{a_1, a_2, \dots, a_2, a_1, 2a_0}]$.

QED ∎