MA 414 Project

create_zeta_inverse calculates the inverse of the Ihara zeta function for a graph G. We use the fact that the degree matrix equals the adjacency matrix plus the laplacian matirx.

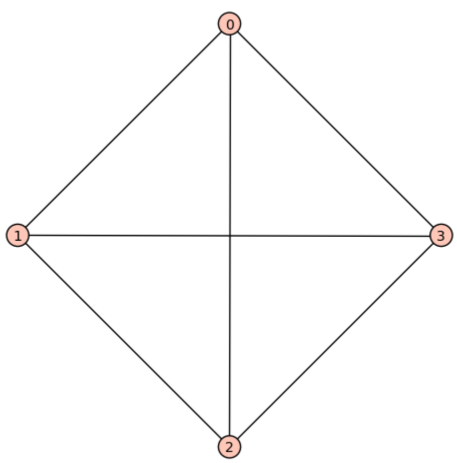
```
def create_zeta_inverse(g):
    var('u')
    Id = identity_matrix(QQ,g.num_verts())
    A = g.adjacency_matrix()
    Q = A + g.laplacian_matrix() - Id
    e = g.num_edges(); n = g.num_verts()
    zi(u) = (1-u^2)^(e - n) * (Id - A*u + Q*u^2).determinant()
    return zi
```

Function rh checks if a graph is regular, and if so, decides whether it is Ramanjun, and whether its Ihara zeta function satisfies the Riemann hypothesis for graphs. These two facts should coincide.

I had a problem with some graphs testing for Re(s) = 1/2, which worked if I used > .50000001 and < .4999999. I suspect it was a roundoff error, though I know that this was a poor workaround.

```
def rh(g):
    q = G.degree_sequence()[0]
    failed_rh = 0
    mu = 0
    if G.is_regular():
        print "Graph is REGULAR with degree", q
        if q < 2: print "Not considering graphs of degree less than 2"; return
        if q == 2: print "Riemann hypothesis vacuously satisfied for 2-regular since 0 <= re(s) <= 1 => re(s)=1/2."; return
        for i in G.spectrum():
            if mu < i.abs() < q: mu = i.abs()
        print "Mu (greatest absolute value of eigenvalue not equal to degree) is", mu
        if mu <= 2*sqrt(q-1): print "G is Ramanujan; WE EXPECT THE RIEMANN HYPOTHESIS TO BE SATISFIED."
        else: print "G is not Ramanujan; WE DO NOT EXPECT THE RIEMANN HYPOTHESIS TO BE SATISFIED"
        zeta_inverse(u) = create_zeta_inverse(g)
        print "Zeta inverse is:"; print zeta_inverse
        print "The roots of zeta inverse with multiplicity are:";
        for i in zeta_inverse.roots(): print "   ", i;
        print "The real root parts are:"
        for i in zeta_inverse.roots():
            q_to_minus_s = i[0]; s = -log(q_to_minus_s,q-1); print "   ", real(s)
        print "0 < Re(s) < 1 for:"
        for i in zeta_inverse.roots():
            q_to_minus_s = i[0]; s = -log(q_to_minus_s,q-1)
            if 0 < real(s) < 1: print "   0 <", real(s), "< 1"
        for i in zeta_inverse.roots():
            q_to_minus_s = i[0]; s = -log(q_to_minus_s,q-1)
            if 0 < real(s) < 1:
                if real(s) > .50000001: failed_rh = 1; print "   FAILED to satisfy RH:", real(s), "has real part > 1/2."
                if real(s) < .49999999: failed_rh = 1; print "   FAILED to satisfy RH:", real(s), "has real part < 1/2."
        if (failed_rh==0): print "THIS GRAPH SATISFIES THE RIEMANN HYPOTHESIS"
        else: print "THIS GRAPH FAILS TO SATISFY THE RIEMANN HYPOTHESIS"
    else: print "Graph is NOT REGULAR; not testing the Riemann Hypothesis."
```

```
G = graphs.CompleteGraph(4)
G.plot()
```



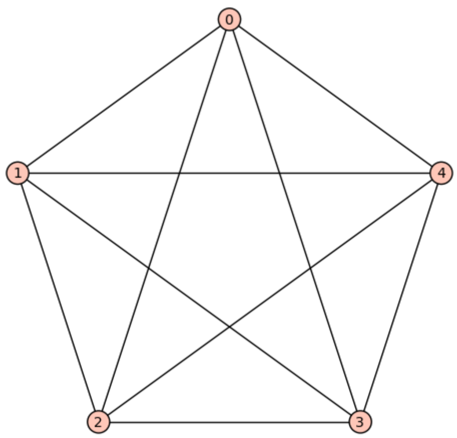

```
rh(G)
```

```
Graph is REGULAR with degree 3
Mu (greatest absolute value of eigenvalue not equal to degree) is 1
G is Ramanujan; WE EXPECT THE RIEMANN HYPOTHESIS TO BE SATISFIED.
Zeta inverse is:
u |--> (u^2 - 1)^2*(16*u^8 + 8*u^6 - 16*u^5 - 3*u^4 - 8*u^3 +
2*u^2 + 1)
The roots of zeta inverse with multiplicity are:
    (1, 3)
    (1/2, 1)
    (-1/4*I*sqrt(7) - 1/4, 3)
    (1/4*I*sqrt(7) - 1/4, 3)
    (-1, 2)
The real root parts are:
    0
    -log(1/2)/log(2)
    -log(abs(-1/4*I*sqrt(7) - 1/4))/log(2)
    -log(abs(1/4*I*sqrt(7) - 1/4))/log(2)
    0
0 < Re(s) < 1 for:
   0 < -log(abs(-1/4*I*sqrt(7) - 1/4))/log(2) < 1
   0 < -log(abs(1/4*I*sqrt(7) - 1/4))/log(2) < 1
THIS GRAPH SATISFIES THE RIEMANN HYPOTHESIS
```

In this case mu is 1, which clearly satisfies mu \leq 2(sqrt(2)), so G is Ramanujan.

Also, the zeta for G has two roots q^(-s) with 0 < real(s) < 1. Both of these roots have real part 1/2, which satisfies the Riemann Hypothesis, as expected, since G is regular and Ramanujan.

```
G = graphs.CompleteGraph(5)
G.plot()
```



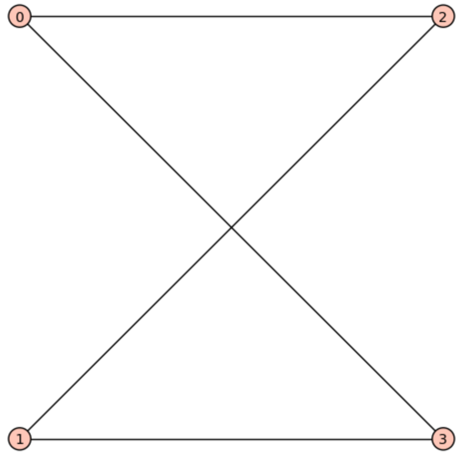

```
rh(G)
```

```
Graph is REGULAR with degree 4
Mu (greatest absolute value of eigenvalue not equal to degree) is 1
G is Ramanujan; WE EXPECT THE RIEMANN HYPOTHESIS TO BE SATISFIED.
Zeta inverse is:
u |--> -(u^2 - 1)^5*(243*u^10 + 135*u^8 - 180*u^7 - 45*u^6 -
124*u^5 - 15*u^4 - 20*u^3 + 5*u^2 + 1)
The roots of zeta inverse with multiplicity are:
    (1, 6)
    (1/3, 1)
    (-1/6*I*sqrt(11) - 1/6, 4)
    (1/6*I*sqrt(11) - 1/6, 4)
    (-1, 5)
The real root parts are:
    0
    -log(1/3)/log(3)
    -log(abs(-1/6*I*sqrt(11) - 1/6))/log(3)
    -log(abs(1/6*I*sqrt(11) - 1/6))/log(3)
    0
0 < Re(s) < 1 for:
   0 < -log(abs(-1/6*I*sqrt(11) - 1/6))/log(3) < 1
   0 < -log(abs(1/6*I*sqrt(11) - 1/6))/log(3) < 1
THIS GRAPH SATISFIES THE RIEMANN HYPOTHESIS
```

```
G = graphs.CompleteBipartiteGraph(5,6)
rh(G)
```

```
Graph is NOT REGULAR; not testing the Riemann Hypothesis.
```

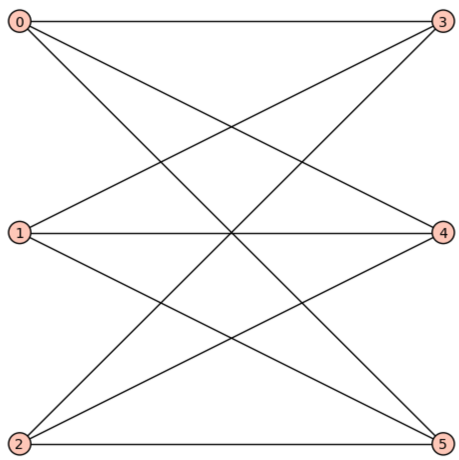
```
G = graphs.CompleteBipartiteGraph(2,2)
G.plot()
```




```
rh(G)
```

```
Graph is REGULAR with degree 2
Riemann hypothesis vacuously satisfied for 2-regular since 0 <=
re(s) <= 1 => re(s)=1/2.
```

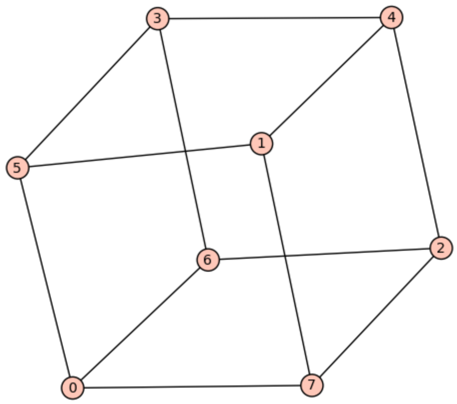
```
G = graphs.CompleteBipartiteGraph(3,3)
G.plot()
```




```
rh(G)
```

```
Graph is REGULAR with degree 3
Mu (greatest absolute value of eigenvalue not equal to degree) is 0
G is Ramanujan; WE EXPECT THE RIEMANN HYPOTHESIS TO BE SATISFIED.
Zeta inverse is:
u |--> -(u^2 - 1)^3*(64*u^12 + 48*u^10 - 48*u^8 - 56*u^6 - 12*u^4
+ 3*u^2 + 1)
The roots of zeta inverse with multiplicity are:
    (-1/2*I*sqrt(2), 4)
    (1/2*I*sqrt(2), 4)
    (-1/2, 1)
    (1/2, 1)
    (-1, 4)
    (1, 4)
The real root parts are:
    -log(abs(-1/2*I*sqrt(2)))/log(2)
    -log(abs(1/2*I*sqrt(2)))/log(2)
    -log(1/2)/log(2)
    -log(1/2)/log(2)
    0
    0
0 < Re(s) < 1 for:
   0 < -log(abs(-1/2*I*sqrt(2)))/log(2) < 1
   0 < -log(abs(1/2*I*sqrt(2)))/log(2) < 1
THIS GRAPH SATISFIES THE RIEMANN HYPOTHESIS
```

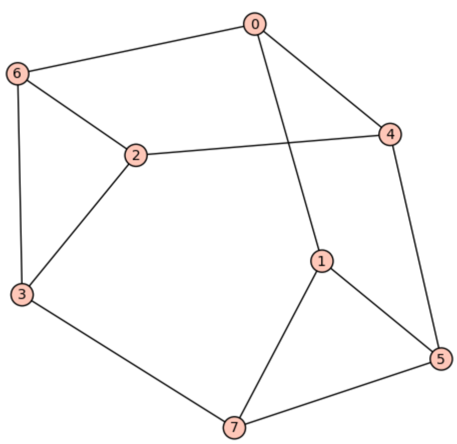
```
G = graphs.RandomRegular(3,8)
G.plot()
```




```
rh(G)
```

```
Graph is REGULAR with degree 3
Mu (greatest absolute value of eigenvalue not equal to degree) is 1
G is Ramanujan; WE EXPECT THE RIEMANN HYPOTHESIS TO BE SATISFIED.
Zeta inverse is:
u |--> (u^2 - 1)^4*(256*u^16 + 256*u^14 - 32*u^12 - 240*u^10 -
183*u^8 - 60*u^6 - 2*u^4 + 4*u^2 + 1)
The roots of zeta inverse with multiplicity are:
    (-1/4*sqrt(I*sqrt(7) - 3)*sqrt(2), 3)
    (1/4*sqrt(I*sqrt(7) - 3)*sqrt(2), 3)
    (-1/4*sqrt(-I*sqrt(7) - 3)*sqrt(2), 3)
    (1/4*sqrt(-I*sqrt(7) - 3)*sqrt(2), 3)
    (-1/2, 1)
    (1/2, 1)
    (-1, 5)
    (1, 5)
The real root parts are:
    -log(abs(-1/4*sqrt(I*sqrt(7) - 3)*sqrt(2)))/log(2)
    -log(abs(1/4*sqrt(I*sqrt(7) - 3)*sqrt(2)))/log(2)
    -log(abs(-1/4*sqrt(-I*sqrt(7) - 3)*sqrt(2)))/log(2)
    -log(abs(1/4*sqrt(-I*sqrt(7) - 3)*sqrt(2)))/log(2)
    -log(1/2)/log(2)
    -log(1/2)/log(2)
    0
    0
0 < Re(s) < 1 for:
   0 < -log(abs(-1/4*sqrt(I*sqrt(7) - 3)*sqrt(2)))/log(2) < 1
   0 < -log(abs(1/4*sqrt(I*sqrt(7) - 3)*sqrt(2)))/log(2) < 1
   0 < -log(abs(-1/4*sqrt(-I*sqrt(7) - 3)*sqrt(2)))/log(2) < 1
   0 < -log(abs(1/4*sqrt(-I*sqrt(7) - 3)*sqrt(2)))/log(2) < 1
THIS GRAPH SATISFIES THE RIEMANN HYPOTHESIS
```

```
G = graphs.RandomRegular(3,8)
G.plot()
```

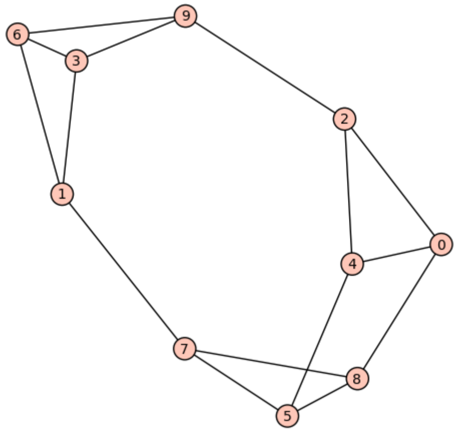



```
rh(G)
```

```
Graph is REGULAR with degree 3
Mu (greatest absolute value of eigenvalue not equal to degree) is
2.414213562373095?
G is Ramanujan; WE EXPECT THE RIEMANN HYPOTHESIS TO BE SATISFIED.
Zeta inverse is:
u |--> (u^2 - 1)^4*(256*u^16 + 256*u^14 - 128*u^13 + 96*u^12 -
192*u^11 - 16*u^10 - 152*u^9 - 23*u^8 - 76*u^7 - 4*u^6 - 24*u^5 +
6*u^4 - 4*u^3 + 4*u^2 + 1)
The roots of zeta inverse with multiplicity are:
    (1, 5)
    (1/2, 1)
    (-1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) - 1/4*sqrt(2) - 1/4, 1)
    (1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) - 1/4*sqrt(2) - 1/4, 1)
    (-1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) + 1/4*sqrt(2) - 1/4, 1)
    (1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) + 1/4*sqrt(2) - 1/4, 1)
    (-1/4*sqrt(I*sqrt(15) - 1)*sqrt(2), 1)
    (1/4*sqrt(I*sqrt(15) - 1)*sqrt(2), 1)
    (-1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2), 1)
    (1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2), 1)
    (-1/4*I*sqrt(7) + 1/4, 1)
    (1/4*I*sqrt(7) + 1/4, 1)
    (-1/4*I*sqrt(7) - 1/4, 2)
    (1/4*I*sqrt(7) - 1/4, 2)
    (-1, 4)
The real root parts are:
    0
    -log(1/2)/log(2)
    -log(abs(-1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) - 1/4*sqrt(2) -
1/4))/log(2)
    -log(abs(1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) - 1/4*sqrt(2) -
1/4))/log(2)
    -log(abs(-1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) + 1/4*sqrt(2) -
1/4))/log(2)
    -log(abs(1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) + 1/4*sqrt(2) -
1/4))/log(2)
    -log(abs(-1/4*sqrt(I*sqrt(15) - 1)*sqrt(2)))/log(2)
    -log(abs(1/4*sqrt(I*sqrt(15) - 1)*sqrt(2)))/log(2)
    -log(abs(-1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2)))/log(2)
    -log(abs(1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2)))/log(2)
    -log(abs(-1/4*I*sqrt(7) + 1/4))/log(2)
    -log(abs(1/4*I*sqrt(7) + 1/4))/log(2)
    -log(abs(-1/4*I*sqrt(7) - 1/4))/log(2)
    -log(abs(1/4*I*sqrt(7) - 1/4))/log(2)
    0
0 < Re(s) < 1 for:
   0 < -log(abs(-1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) - 1/4*sqrt(2)
- 1/4))/log(2) < 1
   0 < -log(abs(1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) - 1/4*sqrt(2) -
1/4))/log(2) < 1
   0 < -log(abs(-1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) + 1/4*sqrt(2)
- 1/4))/log(2) < 1
   0 < -log(abs(1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) + 1/4*sqrt(2) -
1/4))/log(2) < 1
   0 < -log(abs(-1/4*sqrt(I*sqrt(15) - 1)*sqrt(2)))/log(2) < 1
   0 < -log(abs(1/4*sqrt(I*sqrt(15) - 1)*sqrt(2)))/log(2) < 1
   0 < -log(abs(-1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2)))/log(2) <
1
   0 < -log(abs(1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2)))/log(2) < 1
   0 < -log(abs(-1/4*I*sqrt(7) + 1/4))/log(2) < 1
   0 < -log(abs(1/4*I*sqrt(7) + 1/4))/log(2) < 1
   0 < -log(abs(-1/4*I*sqrt(7) - 1/4))/log(2) < 1
   0 < -log(abs(1/4*I*sqrt(7) - 1/4))/log(2) < 1
THIS GRAPH SATISFIES THE RIEMANN HYPOTHESIS
```

```
G = graphs.RandomRegular(3,10)
G.plot()
```

MA 414 Project -- Sage

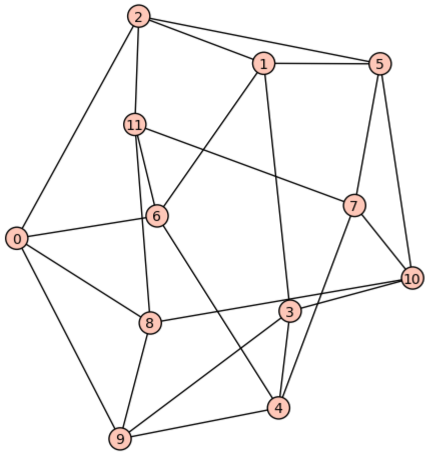



```
rh(G)
```

```
Graph is REGULAR with degree 3
Mu (greatest absolute value of eigenvalue not equal to degree) is
2.414213562373095?
G is Ramanujan; WE EXPECT THE RIEMANN HYPOTHESIS TO BE SATISFIED.
Zeta inverse is:
u |--> -(u^2 - 1)^5*(1024*u^20 + 1280*u^18 - 1024*u^17 + 704*u^16
- 1408*u^15 + 624*u^14 - 992*u^13 + 480*u^12 - 624*u^11 + 248*u^10 -
312*u^9 + 120*u^8 - 124*u^7 + 39*u^6 - 44*u^5 + 11*u^4 - 8*u^3 +
5*u^2 + 1)
The roots of zeta inverse with multiplicity are:
    (1/2, 1)
    (1, 6)
    (-1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) - 1/4*sqrt(2) + 1/4, 1)
    (1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) - 1/4*sqrt(2) + 1/4, 1)
    (-1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) + 1/4*sqrt(2) + 1/4, 1)
    (1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) + 1/4*sqrt(2) + 1/4, 1)
    (-1/4*sqrt(I*sqrt(15) - 1)*sqrt(2), 1)
    (1/4*sqrt(I*sqrt(15) - 1)*sqrt(2), 1)
    (-1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2), 1)
    (1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2), 1)
    (-1/4*I*sqrt(7) - 1/4, 1)
    (1/4*I*sqrt(7) - 1/4, 1)
    (-1/2*I*sqrt(2), 2)
    (1/2*I*sqrt(2), 2)
    (-1/2*I - 1/2, 2)
    (1/2*I - 1/2, 2)
    (-1, 5)
The real root parts are:
    -log(1/2)/log(2)
    0
    -log(abs(-1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) - 1/4*sqrt(2) +
1/4))/log(2)
    -log(abs(1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) - 1/4*sqrt(2) +
1/4))/log(2)
    -log(abs(-1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) + 1/4*sqrt(2) +
1/4))/log(2)
    -log(abs(1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) + 1/4*sqrt(2) +
1/4))/log(2)
    -log(abs(-1/4*sqrt(I*sqrt(15) - 1)*sqrt(2)))/log(2)
    -log(abs(1/4*sqrt(I*sqrt(15) - 1)*sqrt(2)))/log(2)
    -log(abs(-1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2)))/log(2)
    -log(abs(1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2)))/log(2)
    -log(abs(-1/4*I*sqrt(7) - 1/4))/log(2)
    -log(abs(1/4*I*sqrt(7) - 1/4))/log(2)
    -log(abs(-1/2*I*sqrt(2)))/log(2)
    -log(abs(1/2*I*sqrt(2)))/log(2)
    -log(sqrt(1/2))/log(2)
    -log(sqrt(1/2))/log(2)
    0
0 < Re(s) < 1 for:
   0 < -log(abs(-1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) - 1/4*sqrt(2)
+ 1/4))/log(2) < 1
   0 < -log(abs(1/8*I*sqrt(5*sqrt(2) + 4)*2^(3/4) - 1/4*sqrt(2) +
1/4))/log(2) < 1
   0 < -log(abs(-1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) + 1/4*sqrt(2)
+ 1/4))/log(2) < 1
   0 < -log(abs(1/8*I*sqrt(5*sqrt(2) - 4)*2^(3/4) + 1/4*sqrt(2) +
1/4))/log(2) < 1
   0 < -log(abs(-1/4*sqrt(I*sqrt(15) - 1)*sqrt(2)))/log(2) < 1
   0 < -log(abs(1/4*sqrt(I*sqrt(15) - 1)*sqrt(2)))/log(2) < 1
   0 < -log(abs(-1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2)))/log(2) <
1
   0 < -log(abs(1/4*sqrt(-I*sqrt(15) - 1)*sqrt(2)))/log(2) < 1
   0 < -log(abs(-1/4*I*sqrt(7) - 1/4))/log(2) < 1
   0 < -log(abs(1/4*I*sqrt(7) - 1/4))/log(2) < 1
   0 < -log(abs(-1/2*I*sqrt(2)))/log(2) < 1
   0 < -log(abs(1/2*I*sqrt(2)))/log(2) < 1
   0 < -log(sqrt(1/2))/log(2) < 1
   0 < -log(sqrt(1/2))/log(2) < 1
THIS GRAPH SATISFIES THE RIEMANN HYPOTHESIS
```

```
G = graphs.RandomRegular(4,12)
G.plot()
```

MA 414 Project -- Sage




```
rh(G)
```

```
Graph is REGULAR with degree 4
Mu (greatest absolute value of eigenvalue not equal to degree) is
3.257525959723204?
G is Ramanujan; WE EXPECT THE RIEMANN HYPOTHESIS TO BE SATISFIED.
Zeta inverse is:
u |--> (u^2 - 1)^12*(531441*u^24 + 708588*u^22 - 157464*u^21 +
341172*u^20 - 323676*u^19 - 49572*u^18 - 331452*u^17 - 172044*u^16 -
217512*u^15 - 120852*u^14 - 100080*u^13 - 50058*u^12 - 33360*u^11 -
13428*u^10 - 8056*u^9 - 2124*u^8 - 1364*u^7 - 68*u^6 - 148*u^5 +
52*u^4 - 8*u^3 + 12*u^2 + 1)
The roots of zeta inverse with multiplicity are:
    (1/3, 1)
    (1, 13)
    (-1/6*I*sqrt(11) - 1/6, 1)
    (1/6*I*sqrt(11) - 1/6, 1)
    (-1/6*I*sqrt(11) + 1/6, 1)
    (1/6*I*sqrt(11) + 1/6, 1)
    (-1/6*sqrt(I*sqrt(3) - 1)*sqrt(6), 1)
    (1/6*sqrt(I*sqrt(3) - 1)*sqrt(6), 1)
    (-1/6*sqrt(-I*sqrt(3) - 1)*sqrt(6), 1)
    (1/6*sqrt(-I*sqrt(3) - 1)*sqrt(6), 1)
    (-1/3*I*sqrt(3), 1)
    (1/3*I*sqrt(3), 1)
    (-1, 12)
The real root parts are:
    -log(1/3)/log(3)
    0
    -log(abs(-1/6*I*sqrt(11) - 1/6))/log(3)
    -log(abs(1/6*I*sqrt(11) - 1/6))/log(3)
    -log(abs(-1/6*I*sqrt(11) + 1/6))/log(3)
    -log(abs(1/6*I*sqrt(11) + 1/6))/log(3)
    -log(abs(-1/6*sqrt(I*sqrt(3) - 1)*sqrt(6)))/log(3)
    -log(abs(1/6*sqrt(I*sqrt(3) - 1)*sqrt(6)))/log(3)
    -log(abs(-1/6*sqrt(-I*sqrt(3) - 1)*sqrt(6)))/log(3)
    -log(abs(1/6*sqrt(-I*sqrt(3) - 1)*sqrt(6)))/log(3)
    -log(abs(-1/3*I*sqrt(3)))/log(3)
    -log(abs(1/3*I*sqrt(3)))/log(3)
    0
0 < Re(s) < 1 for:
   0 < -log(abs(-1/6*I*sqrt(11) - 1/6))/log(3) < 1
   0 < -log(abs(1/6*I*sqrt(11) - 1/6))/log(3) < 1
   0 < -log(abs(-1/6*I*sqrt(11) + 1/6))/log(3) < 1
   0 < -log(abs(1/6*I*sqrt(11) + 1/6))/log(3) < 1
   0 < -log(abs(-1/6*sqrt(I*sqrt(3) - 1)*sqrt(6)))/log(3) < 1
   0 < -log(abs(1/6*sqrt(I*sqrt(3) - 1)*sqrt(6)))/log(3) < 1
   0 < -log(abs(-1/6*sqrt(-I*sqrt(3) - 1)*sqrt(6)))/log(3) < 1
   0 < -log(abs(1/6*sqrt(-I*sqrt(3) - 1)*sqrt(6)))/log(3) < 1
   0 < -log(abs(-1/3*I*sqrt(3)))/log(3) < 1
   0 < -log(abs(1/3*I*sqrt(3)))/log(3) < 1
THIS GRAPH SATISFIES THE RIEMANN HYPOTHESIS
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