# Exercise Set 2:
# **More Prime Numbers**

Math 414, Winter 2010, University of Washington

Due Friday (!!), January 22, 2010

1. If $f(x)$ and $g(x)$ are (nonzero) functions, we write $f(x) \sim g(x)$ to mean that $\lim_{x\to\infty} f(x)/g(x) = 1$. Prove that for any real number $a$ we have
$$\frac{\log(x)}{x} \sim \frac{\log(x)}{x-a}.$$

2. For any polynomial $f(x)$ with integer coefficients, let
$$P(f) = \{f(n) : n \in \mathbf{N} \text{ and } f(n) \text{ is prime}\}.$$
For example,
$$P(x^2+1) = \{2, 5, 17, 37, 101, 197, \ldots\}.$$
Come up with a *guess* for a condition on $f$ that is equivalent to $P(f)$ containing infinitely many prime numbers. Give evidence for your guess. [Do not worry at all about trying to prove that your guess is correct.] Here is some potentially helpful Sage code, which illustrates computing the number of prime values$f(n)$ for $f = x^3 + x - 1$ and $1 \le n \le 100$:
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sage: f(x) = x^3 + x - 1
sage: len([n for n in [1..100] if is_prime(f(n))])
27
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3. Let $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \ldots$ be the sequence of prime numbers. A *prime triplet* is three consecutive primes $p_n, p_{n+1}, p_{n+2}$ such that $p_{n+1} = p_n + 2$ and $p_{n+2} = p_{n+1} + 2$, i.e., $p_n, p_n + 2, p_n + 4$ are all prime. For example, $3, 5, 7$ are prime triplets. Prove that there are no other prime triplets. [Hint: Suppose $p > 3$ is a prime and you divide $p$ by 12 and take the remainder $r$. Then what are the possibilities for $r$?]