Exercise Set 2 Solutions:
More Prime Numbers

Math 414, Winter 2010, University of Washington

Due Friday (!!), January 22, 2010

1. If $f(x)$ and $g(x)$ are (nonzero) functions, we write $f(x) \sim g(x)$ to mean that $\lim_{x\to\infty} f(x)/g(x) = 1$. Prove that for any real number a we have

$$\frac{\log(x)}{x} \sim \frac{\log(x)}{x-a}.$$

Answer: Taking the quotient this is equivalent to showing that $\lim_{x\to\infty}(x-a)/x = 1$, *which follows from L'Hôpital's rule or just dividing top and bottom by* x.

2. For any polynomial $f(x)$ with integer coefficients, let

$$P(f) = \{f(n) : n \in \mathbf{N} \text{ and } f(n) \text{ is prime}\}.$$

For example,

$$P(x^2+1) = \{2, 5, 17, 37, 101, 197, \ldots\}.$$

Come up with a *guess* for a condition on f that is equivalent to $P(f)$ containing infinitely many prime numbers. Give evidence for your guess. [Do not worry at all about trying to prove that your guess is correct.] Here is some potentially helpful Sage code, which illustrates computing the number of prime values$f(n)$ for $f = x^3 + x - 1$ and $1 \le n \le 100$:

```
sage: f(x) = x^3 + x - 1
sage: len([n for n in [1..100] if is_prime(f(n))])
27
```

Answer: For degree 1, this is Dirichlet's theorem, which says that $f(x) = ax+b$ *takes on infinitely prime values if and only if* $\gcd(a,b) = 1$*, i.e., if only if the polynomial* $f(x)$ *does not factor over* $\mathbb{Z}$*. For degree* ≥ 2*, we similarly conjecture that* f *takes on infinitely many primes if it does not factor over* $\mathbb{Z}$*.*

3. Let $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \ldots$ be the sequence of prime numbers. A *prime triplet* is three consecutive primes p_n, p_{n+1}, p_{n+2} such that $p_{n+1} = p_n + 2$ and $p_{n+2} = p_{n+1} + 2$, i.e., $p_n, p_n + 2, p_n + 4$ are all prime. For example, $3, 5, 7$ are prime triplets. Prove that there are no other prime triplets. [Hint: Suppose $p > 3$ is a prime and you divide p by 12 and take the remainder r. Then what are the possibilities for r?]

   Answer: One of the numbers $p, p+2, p+4$ *must be divisible by* 3*, since* $0, 2, 4$ *represent all numbers mod* 3*. The only prime divisible by* 3 *is the prime* 3*, so we must have* $p = 3$*, as claimed.*

   Alternatively, for $p \geq 5$ *we have* $p \pmod{12}$ *is one of* $1, 5, 7, 11$*, since it can't be anything else or* p *wouldn't be prime. But then* $p, p+2, p+4$ *have residue mod 12 in* $\{1, 5, 7, 11\}$*. But a simple inspection shows that this is impossible for each of* $p \equiv 1, 5, 7, 11 \pmod{12}$*.*