The Modular Group and Klein's Absolute Invariant

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Abstract

The proof of Fermat's Last Theorem stronlgy relies on the connection between elliptic curves and modular functions. In 1984, in fact, Professor Frey showed how a solution to Fermat's equation would give rise to a semi-stable elliptic curve which would not be modular, violating the Tanyiama-Shimura conjecture. The purpose of this paper is to explore basic concepts about modular forms and to define the modular function $J(\tau)$, from which any modular function can be derived.

1 Mobius Transformations in $\hat{\mathbb{C}}$

1.1 The Riemann Sphere

Definition 1. *A "line" in $\mathbb{C}$ is the locus of points z satisfying the equation:*

$$|z-\alpha| = |z-\beta| \tag{1}$$

where z is a complex variable and $\alpha, \beta \in \mathbb{C}$

Geometrically (Fig. 1) , we define a line to be the set of points in the complex plane which are equidistant from two fixed points α, β:

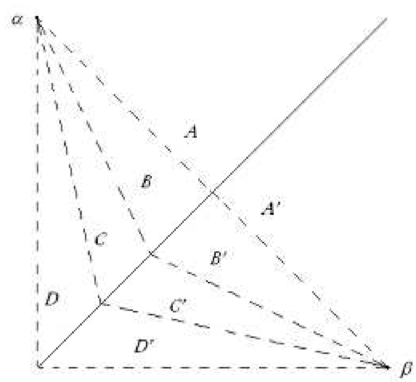


Fig. 1

Each of the segments A, B, C, D has the same length of the segments A', B', C', D' so that a unique line can be drawn that connects the points of intersection of all such segments.

Dividing both sides of equation (1) by $|z-\beta|$ we get:

$$\frac{|z-\alpha|}{|z-\beta|} = 1$$

Now, if we substitute the constant 1 with a parameter, call it λ, which takes positive real values other than 1, our original line becomes a circle. In particular, the circle is defined as the locus of points whose distances from two fixed points are in a constant ratio:

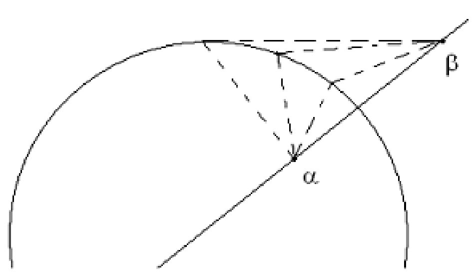


Fig. 2

We can then intepret lines not as a separate geometrical entity, but as a special family of circles where $\lambda = 1$. Such a statement may sound strange to a reader familiar with euclidean geometry, but it can be given a geometrical interpretation by constructing a device that will unify the concept of line and circle in one single geometrical object.

Definition 2. *Consider $\mathbb{C}$ as embedded in $\mathbb{R}^3$, with $z = x + y\,\mathrm{i} \rightarrow (x, y, 0)$. Then, the Riemann Sphere Σ is defined as:*

$$\Sigma := \{(x, y, u) \in \mathbb{R}^3 : x^2 + y^2 + (u - \frac{1}{2})^2 = \frac{1}{4}\}$$

Graphically, the Riemann Sphere is a sphere of diameter 1 touching the complex plane at exactly one point, the origin.

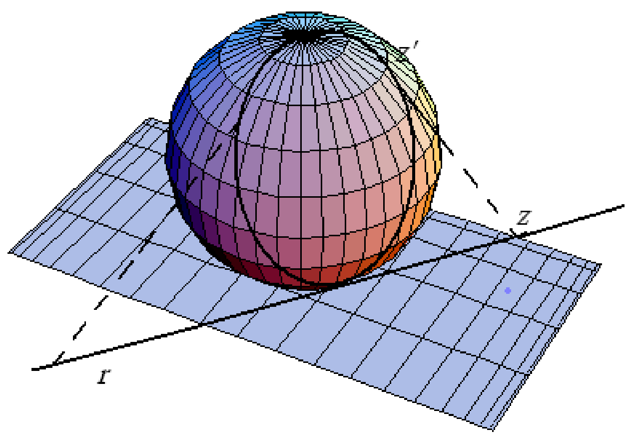


Fig. 3

For any point $z \in \mathbb{C}$, we can draw a line from z to the north pole of the Riemann Sphere, and label z' the point of intersection between such line and the surface of the sphere. In this way, we can establish a 1-1 correspondence between points of $\mathbb{C}$ and points on the surface of the sphere minus the north pole; the origin, in fact, is mapped to itself, since it is itself a point of intersection between the Riemann Sphere and $\mathbb{C}$. In order to avoid this, we map the north pole to the point at infinity, so that, now, every point on the surface of the sphere is in bijection with $\{\mathbb{C}\} \bigcup \{\infty\} = \hat{\mathbb{C}}$.

Now, every circle drawn on the surface of the sphere will be mapped into a circle on the plane, and viceversa. However, if the circle drawn on the surface intersects the north pole, then its projection on the plane will necessarily have an infinite radius (i.e a line). Conversely, all the lines in the complex plane are mapped into circles passing through the north pole. We can then conclude than the set of points $z \in \hat{\mathbb{C}}$ satisfying the equation

$$\frac{|z - \alpha|}{|z - \beta|} = \lambda$$

where $\lambda \in \mathbb{R}^+$, will be mapped into circles on the surface of the Riemann Sphere. From now on, we will refere to these sets as ‘‘circlines”.

1.2 Transformations in $\hat{\mathbb{C}}$

Consider the following transformations:

$$\begin{array}{ll} z \rightarrow z\,\mathrm{e}^{\mathrm{i}\theta}, \theta \in \mathbb{R} & \text{(couter-clockwise rotation through } \theta) \\ z \rightarrow R\,z, \quad R > 0 & \text{(stretching by a factor of R)} \\ z \rightarrow z + a, \ a \in \mathbb{R} & \text{(translation by } a) \end{array}$$

$$z \to 1/z \qquad \text{(inversion)}$$

All of these functions belong to a broader family of transformations called *Mobius Transformations:*

Definition 3. *A Mobius Transformation is a mapping of the form:*

$$z \to w = f(z) := \frac{a\,z + b}{c\,z + d} \qquad (a, b, c, d \in \mathbb{C}, a\,d - b\,c \neq 0)$$

Such tranformations are well-defined over $\hat{\mathbb{C}}$ by letting $f(-d/c) = \infty$ and $f(\infty) = a/c$, according to the geometric interpretation of the exented complex plane that we gave above. We avoid the case where $a\,d - b\,c = 0$, since we would have f constant. In fact, if we denote $f(z)$ and $f(w)$ as two transformations with same coefficients a, b, c, d and $a\,d - b\,c = 0$, then we would have:

$$f(w) - f(z) = \frac{(a\,d - b\,c)\,(w - z)}{(c\,w + d)\,(c\,z + d)} = 0$$

The interesting property of Mobius Transformations is that they map circlines into circlines.

We will now focus our attention on a particular subset of Mobius Transformations, the case when a, b, c, d are integer coefficients and $a\,d - b\,c = 1$. Such set of transformations are known as "the Modular Group Γ" which plays a key-role in the proof of Fermat's Last Theorem.

2 The Modular Group Γ

The equation that appears in Definition 3 is entirely determined by the coefficients a, b, c, d. It is convenient, then, to associate a 2x2 matrix with entries a, b, c, d to each transformation. Call Γ the set of matrices so obtained. It can be shown that Γ forms a group under multiplication.

Theorem 4. Γ *is a group under multiplication*

Proof. It suffices to show that Γ satisfies each of the four group axioms.

1. If $A \in \Gamma$ and $B \in \Gamma$, then $A\,B \in \Gamma$. This is true, since determinants are distributive

$$|\,A\,|\,|\,B\,| = |\,A\,B\,|$$

   Also, since integers are closed under multiplication and addition, the entries of the matrix $A\,B$ will be integers as well.

2. Since matrix multiplication is associative, $(A\,B)\,C = A\,(B\,C)$, with $A, B, C \in \Gamma$

3. The identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a member of Γ

4. For each $A \in \Gamma$, there exists an $A^{-1} \in \Gamma$ such that $A\,A^{-1} = I$. By definition, given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$A^{-1} = \frac{1}{a\,d - b\,c} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

   Since $A \in \Gamma$, $\det A^{-1} = a\,d - b\,c = 1$, so that $A^{-1} \in \Gamma$.

□

From now on, we will refer to Γ interchangeably as both the set of matrices with determinant 1 and integer coefficients and the subset of mobius transformation associated with them. It can also be shown that Γ is generated by two matrices.

Theorem 5. Γ *is generated by the two matrices:*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where every $A \in \Gamma$ can be expressed in the form

$$A = T^{n_1} S T^{n_2} S \ldots S T^{n_k}$$

where n_i are integers, and the representation is not unique.

The proof, which we will not report here, procedes by induction on the element c and by using the fact that $a\,d - b\,c = 1$.

Now, let $\mathbb{H} = \{z \in \mathbb{C}: \mathrm{Im}[z] > 0\}$ (this set, for obvious geometrical reasons, is also called the *upper-half plane*) and define a relation G as following:

For every $z, w \in \mathbb{H}$, $z \backsim w$ iff $\exists$ *an $A \in \Gamma$ such that $z\,A = w$*

The product $z\,A$ can be interpreted as the action on z of the Mobius transformation determined by the coefficients of A. In other words, if we let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $f_A(z) = \frac{a\,z + b}{c\,z + d}$ then $z \backsim w$ if and only if there exists an $A \in \Gamma$ such that $f_A(z) = w$, for every $z \in \mathbb{H}$. Since we showed above that Γ satisfies the group axioms under multiplication, it follows that the relation G is in fact an equivalence relation.

3 The Hyperbolic Upper-Half Plane $\mathbb{H}/\Gamma$

As any equivalence relation does, R creates partitions of $\mathbb{H}$, so that $\mathbb{H}$ can be expressed as a disjoint union of equivalence classes. If we apply these partitions to $\mathbb{H}$, the upper-half plane is transformed into a new geometrical object, which we will call $\mathbb{H}/\Gamma$, the hyperbolic upper-half plane. How does this new monster look like, and what are its propertiesΓ Let's take a look, for example, to what happens to the circline $L_0 = \{z \in \mathbb{H}: \mathrm{Re}[z] = 0\}$, the imaginary axis. By using the real parameter $t > 0$ we can parametrized the axis with $L_0 = \{z \in \mathbb{C}: z = t\,\mathrm{i}, t \in \mathbb{R}_{>0}\}$. Apply the transformation $f_A(z)$ to L_0:

$$f_A(t\,\mathrm{i}) = \frac{a\,t\,\mathrm{i} + b}{c\,t\,\mathrm{i} + d} = \frac{a\,c\,t^2 + b\,d}{c\,t^2 + d^2} + \frac{t}{c^2\,t^2 + d^2}\mathrm{i}$$

On the right-hand side of the equation above we separated the real component from the imaginary component, so that we now have a clear picture of what the circline $f_A(t\,\mathrm{i})$ looks like for a general matrix $A \in \Gamma$. If we consider in fact $f_A(t\,\mathrm{i})$ as a function in $\mathbb{R}^2$, with parametrization $x(t) = \mathrm{Re}[f_A(t\,\mathrm{i})]$ and $y(t) = \mathrm{Im}[f_A(t\,\mathrm{i})]$, $t > 0$, then $f_A(t\,\mathrm{i})$ would look like a semicircle centered on the x-axis. Precisely, as $t \to 0$ we have:

$$\lim_{t \to 0} \mathrm{Re}[f_A(t\,\mathrm{i})] = \frac{b}{d}$$

$$\lim_{t \to 0} \mathrm{Im}[f_A(t\,\mathrm{i})] = 0$$

And if we let $t \to \infty$, we get:

$$\lim_{t \to \infty} \mathrm{Re}[f_A(t\,\mathrm{i})] = a$$

$$\lim_{t \to \infty} \mathrm{Im}[f_A(t\,\mathrm{i})] = 0$$

So that the coordinates of the center of the semicircle given by $f_A(t\ \mathrm{i})$ will be $(\frac{a\,d + b}{2}, 0)$. Moreover, the radius will be given by the maximum value of the function $y(t) = \mathrm{Im}[f_A(t\,\mathrm{i})]$:

$$y'(t) = \frac{c^2\,t^2 + d^2 - 2\,c^2\,t^2}{(c^2\ t^2 + d^2)^2}$$

Equating $y'(t) = 0$ we get:

$$c^2 t^2 = d^2 \Rightarrow t = \frac{d}{c} \Rightarrow y_{\text{Max}} = y(\frac{d}{c}) = \frac{1}{2\,c\,d}$$

Notice how, whenever c or d are 0, the radius of the semicircle can be considered infinite, and the imaginary axis either translates (in case $c = 0$) or is inverted on itself (case when $d = 0$). Also, when the identity matrix is applied to the imaginary axis, as expected, nothing changes.

In a more general setting, every vertical line in the complex plane is transformed by any matrix $A \in \mathbb{C}$ into either a semicircle with center on the real axis or translated to the left or to the right.

Choose now a point z in $\mathbb{H}$. The set of all points $w = f_A(z)$ for $A \in \Gamma$, is called an *orbit*. In other words, an *orbit* is the equivalence class of points $z \in \mathbb{C}$ defined by the relation R defined in the last paragraph of Section 2. In the picture below, that shows the action of Γ on the complex plane, the black spots mark an orbit in $\mathbb{H}/\Gamma$:

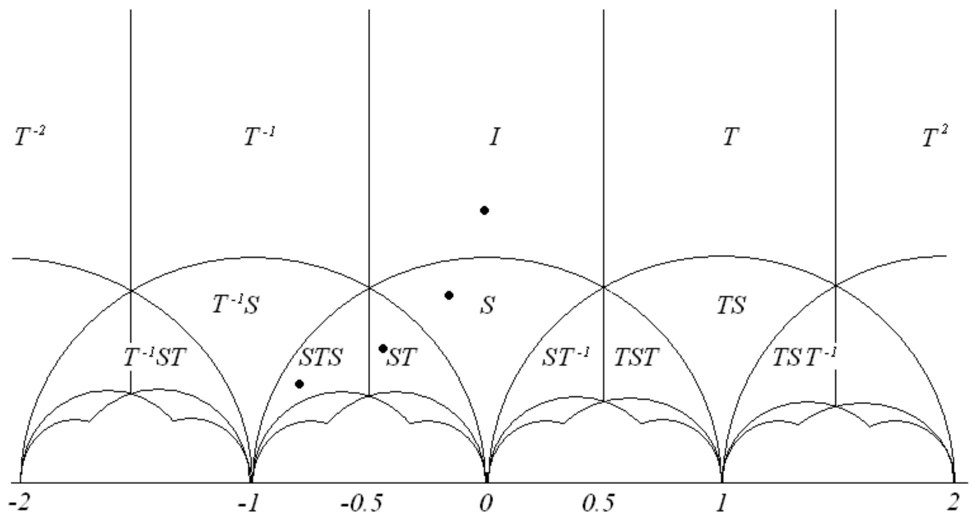


Figure 1.

Keeping in mind the definition of the equivalence relation R, we will now proceed in exploring some properties of $\mathbb{H}/\Gamma$:

Definition 6. *A fundamental region of a subgroup $G \in \Gamma$ is an open set $R_G \subset \mathbb{H}$ with the two following properties:*

a) No two distict points of R_G are equivalent under G

b) If $z \in \mathbb{H}$ there exists a $w \in \overline{R_G}$ (the closure of R_G) such that w is equivalent to z under G

In other words, a fundamental region is a portion of the plane that can be taken as representative of the entire plane, since it contains exaclty one member of each of the equivalence classes that form a partition of the plane itself. In the case of a lattice $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{C}$, with double period ω_1, ω_2 , where the ratio ω_1/ω_2 is non-real, a fundamental region would be the parallelogram bounded by $0, \omega_1, \omega_2, \omega_1 + \omega_2$. In the case of $\mathbb{H}/\Gamma$, the fundamental region can be characterized as follows:

Theorem 7. *The open set*

$$R_\Gamma = \{z \in \mathbb{H}: |z| > 1, |z + \bar{z}| < 1\}$$

is a fundamental region for Γ.

Proof. It suffices to show that R_Γ satisfies the two properties listed in definition 6. Property b) follows from a series of lemma that we will not prove in here. We will show, however, that no two distinct points of R_Γ are equivalent under Γ. Let $z, z' \in R_\Gamma$ be two points equivalent under Γ. Then there exists an $A \in \Gamma$, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $z' = A\,z$. We will first show that $\mathrm{Im}[z'] < \mathrm{Im}[z]$, if $c \neq 0$. Directly from the formula for Mobius transformations we have:

$$\mathrm{Im}[z'] = \frac{\mathrm{Im}[z]}{|c\,z + d|^2}$$

Since $c \neq 0$, $z \in R_\Gamma$ we have:

$$|c\,z + d^2| = (c\,z + d)\,(c\,\bar{z} + d) = c^2\,z\,\bar{z} + c\,d(z + \bar{z}) + d^2 > c^2 - |c\,d| + d^2$$

+Now, if $d = 0$ we have $|c\,z + d^2| > 1$. If $d \neq 0$:

$$c^2 - |c\,d| + d^2 = (|c| - |d|)^2 + |c\,d| \geq |c\,d| \geq 1$$

so that $|c\,z + d|^2 > 1 \;\Rightarrow \mathrm{Im}[z'] < \mathrm{Im}[z]$ whenever $c \neq 0$, for each $z \in R_\Gamma$.

Suppose now z and z' are equivalent points of R_Γ under Γ. Then

$$z' = \frac{a\,z + b}{c\,z + d} \text{ and } z = \frac{d\,z' - b}{-c\,z + a}$$

so that, if $c \neq 0$, we have both $\mathrm{Im}[z'] > \mathrm{Im}[z]$ and $\mathrm{Im}[z] > \mathrm{Im}[z']$. Therefore, $c = 0$ and $a\,d = 1$, $a = d = \pm 1$, and:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} = T^{\pm b}$$

But then b must be 0, since both z, $z' \in R_\Gamma$ (any translation with $b \neq 0$ would map z' outside of the fundamental region) $\Longrightarrow z = z'$.

□

In figure 1, we labeled the fundamental region as I . It follows in fact from theorem 7 that if two points z, z' of the fundamental region are equivalent, then it must be that $z' = I\,z = z$. In other words, the fundamental region is a part of the plane that contains a representative member from each equivalence class. Any point in $\mathbb{C}$ is equivalent to some point in the fundamental region, which is to say, for each $z \in \mathbb{C}$, there exists a $w \in R_\Gamma$ such that $z = A\,w$, for some $A \in \Gamma$. The regions in figure 1 are characterized by such A.

We can now make some observations on the space $\mathbb{H}/\Gamma$. First of all, not all the theorems of euclidean geometry hold on this space. This follows from the fact that Euclid's Fifth Postulate, which characterizes euiclidean geometry, does not hold on $\mathbb{H}/\Gamma$. In figure 2, we show how two lines parallel to line a pass through the same point P, which is not on a.

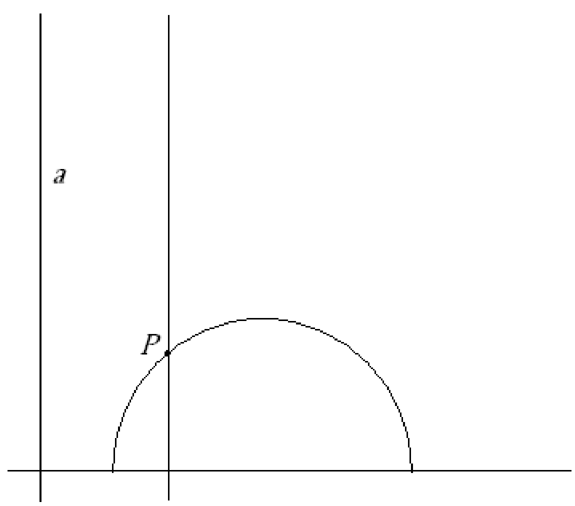
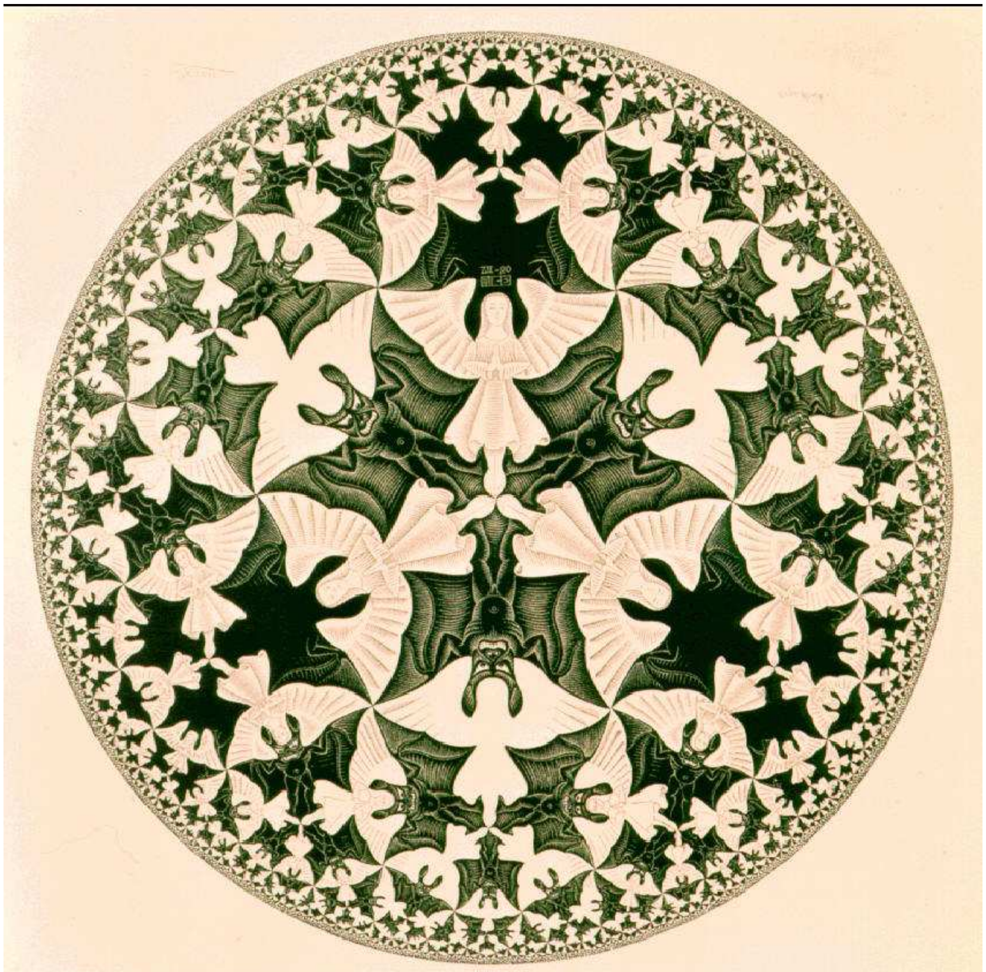


Figure 2.

In fact, there could be infinite lines passing through P and parallel to a. This is why we call $\mathbb{H}/\Gamma$ the hyperbolic upper-half plane, since the axioms of hyperbolic geometry hold on this surface.

The pattern in figure 1 is in fact characteristic of surfaces with hyperbolic geometries. The Poincare' Disk, for example, defined as $D = \{x \in \mathbb{R}^2 \colon |x| < 1\}$ with hyperbolic metric, displays the same pattern of lines and semicircles that violates Euclid's fifth postulate. The drawing *Circle Limit IV* by Escher, shown below, is an artistic interpretation of Poincare's Disk:



It may be hard at first to notice the similarity with figure 1, but imagine to take the real axis and connect the points at $+\infty$ and $-\infty$. Then you would have a circle that contains the upper-half plane, and it becomes clear how the patterns in *Circle Limit IV* can be explained through the symmetries of the Modular Group.

4 Klein's Modular Function $J(\tau)$

We will now focus our attention on a function, $J(\tau)$, which is invariant under any transformation in Γ, and which belongs to a larger class of functions called modular functions. Modular functions are intimately connected with elliptic functions, and the connection between this two different mathematical objects is at the very core of the proof of Fermat's Last Theorem.

We will begin our exploration of the J function by first defining a few basic terms:

Definition 8. *A function f of a complex variable z is called "periodic" with period ω if*

$$f(z+\omega)=f(z)$$

whenever z and $z+\omega$ are in the domain of f.

Suppose we have two periods. ω_1, ω_2 on the complex plane $\mathbb{C}$, where, in order to avoid degenerate cases, the ratio ω_1/ω_2 is non-real. Then every linear combination $\omega = n\,\omega_1 + m\,\omega_2$ is also a period, and we can construct a grid on the complex plane where every corner of each parallelogram is a linear combination of the two periods. Such "grid" is technically called a *lattice* generated by ω. Two pairs of periods (ω_1, ω_2) and (ω_1', ω_2') are called *equivalent* if they generate the same lattice Λ. The following theorem establishes a condition for equivalence of periods:

Theorem 9. *Two pairs of periods (ω_1,ω_2) and (ω_1',ω_2') are equivalent if and only if there exists a matrix $A\in\Gamma$ such that $\omega'=A\,\omega$*

The results of theorem 9, as we will see shortly, play a fundamental role in the construction of the J function. Let's now proceed in describing functions which are doubly periodic on the complex plane. Constant functions are a trivial example of those, but there are also nonconstant meromorphic (a function is meromorphic when its only singularities in the finite plane are poles) functions which are doubly periodic. These are called *elliptic functions*. As it turns out, elliptic functions are an incredible instrument for solving problems in number theory, in particular, for proving Fermat's Last Theorem.

How do these functions look likeΓ We will show here a simplified procedure on how to construct the Weierstrass equation, from which every elliptic function can be derived.

Since ellptic functions are, by definition, holomorphic everywhere but at at poles, they can all be described by a Laurent series expansion. Since we also want the function to be doubly periodic, we may think of the function:

$$f(z)=\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^2}$$

If we assume for a moment that the series above converge absolutely, than we can see why it is periodic. In fact,

$$f(z+\omega_1)=\sum_{\omega\in\Lambda}\frac{1}{(z+\omega_1-\omega)^2}$$

but $(z+\omega_1-\omega)$ is already a member of the lattice, so that the above series can be regarded as a rearrangement of the series defining $f(z)$. Since we assumed that the series converges absolutely, than we conclude that $f(z)=f(z+\omega_1)$, and the same reasoning holds for ω_2.

However, it turns out that the series:

$$\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^n} \tag{2}$$

does not converge absolutely for $n=2$. In fact, it can be shown that it does converge, absolutely and uniformly, for all exponents $n>2$. The proof, however, is omitted here. We then finally obtain our elliptic function by raising the exponent to $n=3$:

$$f(z)=\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3} \tag{3}$$

the above function, in fact, is doubly periodic on the complex plane, and it exhibits 3rd order poles at each ω.

Is it possible to construct an elliptic function with order 2 polesΓ The answer is yes. Since the series in (3) is uniformly convergent, we can integrate it term by term to get:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

(we omitted some techincal details about the integration process). This is called the *Weierstrass* $\wp$ *function*. Its importance relies on the fact that any elliptic function can be expressed as a rational function of $\wp$ and, conversely, any rational function of $\wp$ is an elliptic function. Notice that $\wp$ is an even function since:

$$(-z-\omega)^2 = (z+\omega)^2 = (z-(-\omega))^2$$

but the expression $-\omega$ is equivalent to ω, since they run through the same set of periods. Therefore, $f(z) = f(-z)$.

The function $\wp$ satisfies also the differential equation:

$$\wp'(z)^2 = 4\,\wp(z)^3 - 60\,G_4\,\wp(z) - 140\,G_6 \tag{4}$$

where G_n indicates the *Eisentstein series of order n*, defined as:

$$G_n = \sum_{\omega \in \Lambda} \frac{1}{\omega^n}$$

Conventionally, we let $g_2 = 60\;G_4$ and $g_3 = 140\;G_6$ and we call g_2 and g_3 the *invariants.* In fact, it can be shown that any Eisenstein series is expressible as a polynomial in g_2 and g_3. Equation 4 now takes the form of:

$$\wp'(z)^2 = 4\,\wp(z)^3 - g_2\,\wp(z) - g_3 \tag{5}$$

The right-hand side of equation 5 is a third-degree polynomial in $\wp(z)$, with three distinct solutions $\wp(\omega_1/2)$, $\wp(\omega_2/2)$, $\wp[(\omega_1+\omega_2)/2]$, the haf-periods. Since the three roots are distinct, we conclude that the discriminant of the polynomial must be nonzero, which is to say:

$$\Delta = g_2^3 - 27\,g_3^2 \neq 0$$

Remember, however, that the discriminant, being entirely determined by the invariants g_2, g_3, which are functions of the periods ω_1, ω_2, is itself a function of ω_1, ω_2. In fact, since g_2 and g_3 are homogenous functions by definition, of order -4 and -6 respectively, then it follows that Δ is a homogeneous function of order -12. which is to say, for any $\lambda \neq 0$, we have:

$$\Delta(\lambda\,\omega_1, \lambda\,\omega_2) = \lambda^{-12}\,\Delta(\omega_1, \omega_2)$$

Now, set $\tau = \frac{\omega_1}{\omega_2}$ and let $\lambda = \frac{1}{\omega_1}$. Then it follows that Δ, thorugh a simple change of scale, is a function of one complex variable τ.

$$\Delta(1, \tau) = \omega_1^{12}\,\Delta(\omega_1, \omega_2) \tag{6}$$

We are now ready to define Klein's Modular Function, $J(\tau)$, also known as Klein's Absolute Invariant, which is a combination of g_2 and g_3 such that $J(\tau)$ is homogeneous of degree 0:

$$J(\omega_1, \omega_2) = \frac{g_2^3(\omega_1, \omega_2)}{\Delta(\omega_1, \omega_2)}$$

It follows then that $J(\lambda\,\omega_1,\,\lambda\,\omega_2) = \lambda^0 J(\omega_1,\,\omega) = J(\omega_1,\,\omega_2)$ and, by the same substitution used in 6, we have:

$$J(1, \tau) = J(\omega_1, \omega_2)$$

So that J is, in fact, a function of the ratio τ alone, and we write $J(\tau)$. The key property of the J function is that it is invariant under the action of Γ. Consider in fact the pair of periods $\omega_1, \omega_2,$ and their images ω_1', ω_2' under an arbitrary transformation $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then we have:

$$\tau' = \frac{\omega_2'}{\omega_1'} = \frac{a\,\omega_2 + b\,\omega_1}{c\,\omega_2 + d\,\omega_1} = \frac{a\,\tau + b}{c\,\tau + d}$$

However, from theorem 9 we know that the pairs of periods ω_1, ω_2 and ω_1', ω_2' are equivalent (the originate the same lattice Λ), which implies $J(\omega_1, \omega_2) = J(\omega_1', \omega_2')$. Moreover, we have:

$$\mathrm{Im}[\tau'] = \frac{\mathrm{Im}[\tau]}{|c\,\tau + d|^2}$$

so that, together with the above, we can show how $J(\tau)$ is invariant under the action of Γ.