Parametrization and the equation a^3+b^3=c^2

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1 Introduction

This paper will utilize the process of parametrization to find solutions for the Diophantine equation:

$$a^3+b^3=c^2$$

This Fermat-like equation is an interesting cross between Fermat's Last Theorem for exponent 3 (for which there are no non-trivial solutions) and Fermat's Last Theorem for exponent 2 (which has infinitely many integer solutions). At first glance, it is hard to say whether there exist infinitely many integer solutions a, b, c or not. What are we to expect?

1.1 The Fermat-Catalan Conjecture

It turns out that shortly after Andrew Wiles presented his proof for Fermat's Last Theorem, mathematicians Henri Darmon and Andrew Granville submitted a conjecture that generalizes Fermat's Last Theorem as well as Diophantine equations like $a^3+b^3=c^2$. The conjecture says the equation:

$$a^p+b^q=c^r$$

i. has infinitely many non-trivial integer solutions a, b, c if: $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$

ii. has finitely many integer solutions a, b, c if: $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$

iii. has finitely many integer solutions a, b, c if: $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$

Fermat's Last Theorem satisfies these conditions since there are infinitely many Pythagorean triples (the $n=2$ case), yet there exist no solutions for any $n>2$.

Note that the second case is only satisfied by $(p,q,r)=(2,4,4), (2,3,6), (3,3,3)$ The (3,3,3) case was shown by Euler's proof to have no solutions, and the (2,4,4) case was shown by Fermat to have no solutions. For the (2,3,6) case Darmon and Granville state that no non-trivial proper solutions exist.

There exist a few solutions to the third case. For example, $1+2^3=3^2$ and $2^5+7^2=3^4$. Catalan's Conjecture states that 8 and 9 are the only consecutive non-trivial numbers which are both integer powers of other numbers. This conjecture supports the conjecture that there exist only finitely many solutions for the third case. Darmon and Granville propose that all such solutions have already been found.

From the Fermat-Catalan Conjecture we can anticipate infinitely many solutions for the equation $a^3+b^3=c^2$ since $\frac{1}{3}+\frac{1}{3}+\frac{1}{2}>1$.

1.2 Parametrization

Our goal is to find a way to produce infinitely many solutions for $a^3+b^3=c^2$. The technique of parametrization is useful for this process.

Suppose there is a curve on which we know one point. Is there any way to find another rational or real point of the curve? In some cases, it is possible to use parametrization to achieve the goal. This method involves taking a known point, and connecting it to another point on the curve by drawing a line. To find general solutions, one must express the variables solely in terms of t, the slope of a line connecting the known point to a new point on the curve.

Example 1. Let us examine the equation $x^2-y^2=1$ (E1)

A known point on this curve is $(x, y) = (1, 0)$. Then allow that the line with slope t connects $(1,\ 0)$ with another $(x,\ y)$ on the same curve. From the slope-intercept form of a line, we find that $y = t(x-1)$ (E2)

Substituting (E2) into (E1) we find that:

$$x^2 - t^2(x-1) = 1, \text{ which simplifies to}$$
$$x^2 - t^2x^2 + 2t^2x - t^2 - 1 = 0$$

Next, we group the terms and divide through by the leading coefficient.

$$x^2 + \frac{2t^2}{1-t^2}x + \frac{-t^2-1}{1-t^2} = 0$$

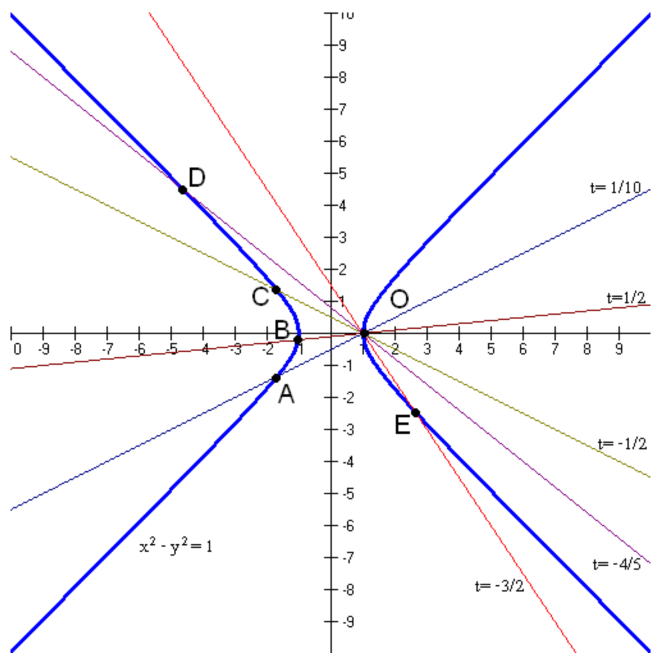
There are two roots to this equation. One is the point we have already found, and the other is a new point determined by t. We factor to find this root.

$$(x-1)\left(x - \frac{t^2+1}{t^2-1}\right) = 0$$

So $x = \frac{t^2+1}{t^2-1}$. Since $y = t(x-1)$, $y = t\left(\frac{t^2+1-t^2+1}{t^2-1}\right) = \frac{2t}{t^{2-1}}$

The parametrized solution for $x^2 - y^2 = 1$ is $(x, y) = \left(\frac{t^2+1}{t^2-1}, \frac{2t}{t^2-1}\right)$

We have reached the goal of finding additional points on the curve, since substituting in any value of t will give a new point (x, y)



Above is a graph of $x^2 - y^2 = 1$, with our known point, $O = (1, 0)$. Drawn from O are lines with various slopes which connect point O to another point on the curve $x^2 - y^2 = 1$. From the parametrization $(x,\ y) = \left(\frac{t^2+1}{t^2-1}, \frac{2t}{t^2-1}\right)$, we can predict and verify the point of intersection of each line. We find that:

Slope	Point of Intersection	Coordinates
t		$\left(\frac{t^2+1}{t^2-1}, \frac{2t}{t^2-1}\right)$
$\frac{1}{10}$	A	$\left(\frac{-101}{99}, \frac{-20}{99}\right)$
$\frac{1}{2}$	B	$\left(\frac{-5}{3}, \frac{-4}{3}\right)$

$-\frac{1}{2}$	C	$(\frac{-5}{3}, \frac{4}{3})$
$-\frac{4}{5}$	D	$(\frac{-41}{9}, \frac{40}{9})$
$-\frac{3}{2}$	E	$(\frac{13}{5}, \frac{-12}{5})$

Each of these points satisfies the equations $x^2 - y^2 = 1$ and $y = t(x-1)$

Suppose we know another point on $x^2 - y^2 = 1$. We can find the slope t that will connect this point to our known point (1, 0) by referring to our parametrized solution. We know that $x = \frac{t^2+1}{t^2-1}$ and $y = \frac{2t}{t^2-1}$. By substituting our new point, we can solve for t.

For example, let's say the point we know to be on $x^2 - y^2 = 1$ to be $(3, 2\sqrt{2})$.

$$3(t^2-1) = t^2+1$$

$$2\sqrt{2}(t^2-1) = 2t$$

So $(3-2\sqrt{2})(t-1)(t+1) = t^2 - 2t + 1 = (t-1)^2$

$$(3-2\sqrt{2})t + (3-2\sqrt{2}) = t-1$$

$$(1-\sqrt{2})t = \sqrt{2}-2$$

$$t = \frac{\sqrt{2}-2}{1-\sqrt{2}}$$

Using parametrization, we can take a quadratic curve on which we know one point and generalize all other points on the curve by drawing a line of slope t to another point on the curve. Using the generalized solution for (x, y), other points on the curve can be found. Also, the slope of a line connecting a new point to our known point can be determined easily from the parametrization.

2 Lemma for Parametrization on the equation $\mathrm{a}^{\mathbf{3}} + \mathrm{b}^{\mathbf{3}} = \mathrm{c}^{\mathbf{2}}$

Jim Buddenhagen presents a proof[1] showing that there exist many solutions to this equation for a and b both prime numbers. He proves the following lemma, which I will utilize to parametrize solutions for a and b

Lemma 2. If $\gcd(a,b) = 1$ and $a^3 + b^3 = c^2$, then either

Case 1: $a+b$ and $a^2 - ab + b^2$ are both squares, or

Case 2: $a+b$ and $a^2 - ab + b^2$ are both three times a square

3 Case 1

3.3 Parametrization of the square factor $\boldsymbol{a^2 - ab + b^2}$

It would be nice if we were able to parametrize the equation from its form of $a^3 + b^3 = c^2$.

Why are we not able to? The problem is that there are three unknowns, and this equation forms a surface rather than a curve. In order to parametrize this equation, we must find a way to parametrize a quadratic curve.

We know that $\boldsymbol{a^2 - ab - b^2}$ is a square. Let this number equal $\boldsymbol{k^2}$. Then:

$$\boldsymbol{a^2 - \mathrm{ab} + b^2 = k^2}$$

Dividing through by $\boldsymbol{k^2}$ yields:

$$\boldsymbol{\frac{a^2}{k^2} - \frac{a}{k}\frac{b}{k} + \frac{b^2}{k^2} =}$$

This can be expressed as an equation with two unknowns by setting $x = \frac{a}{k}$ and $y = \frac{b}{k}$

Now, $x^2 - \mathrm{xy} + y^2 = 1$. Since we have two unknowns in a quadratic equation, we are no longer dealing with a surface, but instead with a curve that we can parametrize with the same technique as the example.

The point $(-1,0)$ is a solution, so $y=t(x+1)$

By substitution, $x^2-\mathrm{tx}^2-\mathrm{tx}+t^2x^2+2\mathrm{xt}^2+t^2-1=0$.

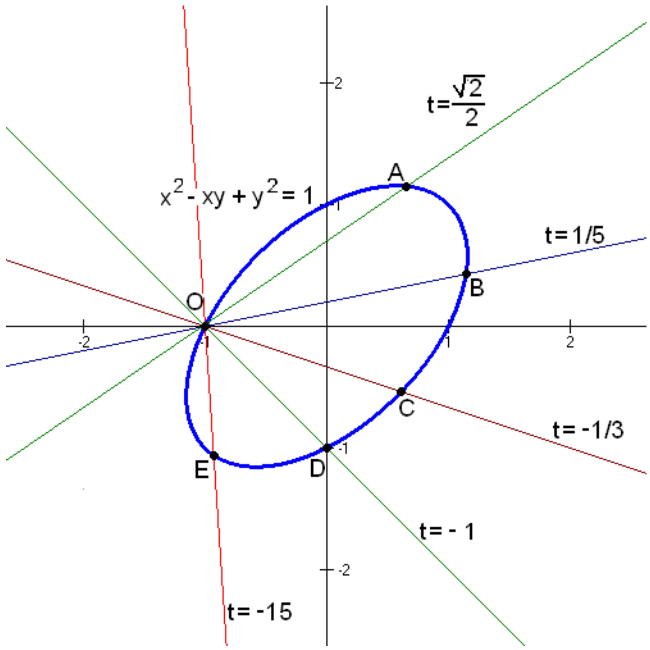
Next, we see $\quad x^2+\frac{2t^2-t}{t^2-t+1}x+\frac{t^2-1}{t^2-t+1}=0$

So $\quad (x+1)(x+\frac{t^2-1}{t^2-t+1})=0$ and $x=\frac{a}{k}=\frac{1-t^2}{t^2-t+1}$

We also find $\quad y=\frac{b}{k}=t(\frac{(1-t^2)+(t^2-t+1)}{t^2-t+1})=\frac{2t-t^2}{t^2-t+1}$

Thus we determine that $a=1-t^2$, $b=2t-t^2$, $k=t^2-t+1$

Below is the graph of $x^2-\mathrm{xy}+y^2=1$:



Lines of various slopes intersect both our known point $(-1, 0)$ and another point on the curve $x^2-\mathrm{xy}+y^2=1$.

Slope	Point of Intersection	Coordinates
t		$(\frac{1-t^2}{t^2-t+1}, \frac{2t-t^2}{t^2-t+1})$
$\frac{\sqrt{2}}{2}$	A	$(\frac{1}{3-\sqrt{2}}, \frac{2\sqrt{2}-1}{3-\sqrt{2}})$
$\frac{1}{5}$	B	$(\frac{8}{7}, \frac{3}{7})$
$-\frac{1}{3}$	C	$(\frac{8}{13}, \frac{-7}{13})$
-1	D	$(0,-1)$
-15	E	$(\frac{-224}{241}, \frac{-255}{241})$

Our parametrization has allowed us to find 5 additional points that satisfy $x^2-\mathrm{xy}+y^2=1$. Yet not all the points we may generate are useful. When $t=-1$, our solution $x=0$, $y=-1$ is not distinct from $x=-1$, $y=0$. The purpose of the parametrization is to aid in finding other integer solutions for $a, b,$ and c. Since $x=\frac{a}{k}$ and $y=\frac{b}{k}$, and k is a factor of c, a rational solution to our parametrization is acceptable. However notice that the solution for b and k when $t=\frac{\sqrt{2}}{2}$ is irrational, which is not useful for finding integer solutions of a, b, c. Thus, to ensure we intercept a rational point (x, y) (which means there are integer solutions for a and b), we must pick a rational value of t. Let $t=\frac{r}{s}$. Now

3.1 Parametrization of the square factor $a^2 - ab + b^2$

$a = \frac{s^2 - r^2}{s^2}$ $\quad b = \frac{2rs - r^2}{s^2}$ $\quad k = \frac{r^2 - rs + s^2}{s^2}$

Since we want integer values for a, b, and k, we can multiply each value by s^2, which preserves the equality between the three.

Finally, $a = s^2 - r^2 = (s - r)(s + r)$ $\quad b = 2rs - r^2 = r(2s - r)$ $\quad k = r^2 - rs + s^2$

3.4 Are a, b, k coprime?

First, we may assume that $\gcd(r, s) = 1$. Since $\frac{r}{s} = t$, if $\gcd(r, s) \neq 1$ then $\frac{r}{s}$ can be reduced so that $\gcd(r, s) = 1$.

To show a, b, k are coprime, we assume there is a prime p which divides all three.
If $p|a$, then either $p|(s - r)$ or $p|(s + r)$

3.4.1 p divides $s - r$

If $p|(s - r)$ then $r \equiv s (\mathrm{mod}\, p)$. Substituting this relation in b shows that $b \equiv s^2 (\mathrm{mod}\, p)$
If p also divides b, then s^2 and also $s \equiv 0 (\mathrm{mod}\, p)$. This would make $r \equiv 0 (\mathrm{mod}\, p)$, and since it has already been stated that $\gcd(r, s) = 1$, p must equal 1. Thus a, b, k are coprime

3.4.2 p divides $s + r$

If $p|(s + r)$, then $s \equiv -r (\mathrm{mod}\, p)$. Substituting $-r$ for s results in $b \equiv -3r^2 (\mathrm{mod}\, p)$ and $k \equiv 3r^2 (\mathrm{mod}\ p)$. The largest value that would divide all three then is 3. Then k would be 3 times a square, which is Case 2. If 3 divides k again, then k would not be 3 times a square. Since $k \equiv 3r^2 (\mathrm{mod}\ p)$, however, r^2 would have to be an odd power of 3, which is not possible since r is an integer. So a, b, k are coprime.

3.5 A second parametrization

We have produced parametrized solutions under the condition that $a^2 - ab + b^2$ is a square. However there is another constraint: $a + b$ must also be a square. To further specify the parametrization we must now consider that $a + b = z^2$ for some z. If we replace a and b with the parametrization in terms of r and s, we have a similar equation.

$$s^2 + 2rs - 2r^2 = z^2$$

We set $x = \frac{s}{z}$ and $y = \frac{r}{z}$

$$x^2 + 2xy - 2y^2 = 1 \qquad (x, y) = (1, 0) \text{ is a solution}$$

$$y = t(x - 1)$$

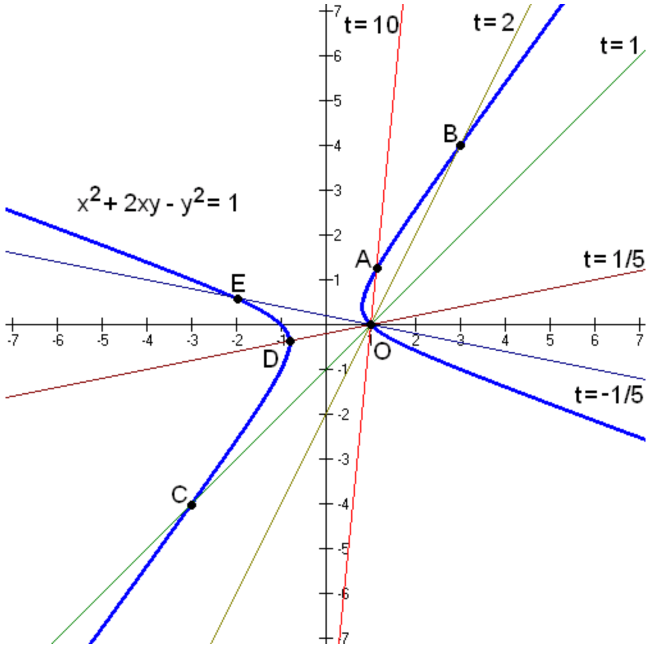
By substitution,
$$x^2 + 2x(tx - t) - 2(tx - t)^2 - 1 = 0$$
$$x^2 + \frac{4t^2 - 2t}{1 + 2t - 2t^2} x + \frac{-2t^2 - 1}{1 + 2t - 2t^2} = 0$$

Since $x = 1$ is a root, $(x - 1)\left(x - \frac{-2t^2 - 1}{1 + 2t - 2t^2}\right) = 0 \qquad x = \frac{2t^2 + 1}{2t^2 - 2t - 1}$

$$y = t\left(\frac{2t^2 + 1}{2t^2 - 2t - 1} - 1\right) = \frac{2t^2 + 2t}{2t^2 - 2t - 1}$$

So the parametrized solution for (x, y) is $\left(\frac{2t^2 + 1}{2t^2 - 2t - 1}, \frac{2t^2 + 2t}{2t^2 - 2t - 1}\right)$

Below is the graph of $x^2 + 2xy - 2y^2 = 1$:



Our known point is $(1, 0)$. We find other parametrized solutions such as:

Slope t	Point of Intersection	Coordinates $\left(\frac{2t^2+1}{2t^2-2t-1}, \frac{2t^2+2t}{2t^2-2t-1}\right)$
10	A	$\left(\frac{201}{179}, \frac{220}{179}\right)$
2	B	$(3, 4)$
1	C	$(-3, -4)$
$\frac{1}{5}$	D	$\left(\frac{-8}{11}, \frac{-4}{13}\right)$
$\frac{-1}{5}$	E	$\left(\frac{-27}{13}, \frac{8}{13}\right)$

The generalized solutions are then:
$s = 2t^2 + 1$ $\quad r = 2t^2 + 2t$ $\quad z = 2t^2 - 2t - 1$

We want integer solutions for r and s, so we set $t = \frac{m}{n}$ and multiply by n^2 to get integers
So $\quad s = 2m^2 + n^2 \quad r = 2m^2 + 2\mathrm{mn} \quad z = 2m^2 - 2\mathrm{mn} - n^2$

3.6 $\boldsymbol{a, b, c}$ in terms of $\boldsymbol{m}$ and $\boldsymbol{n}$

Now, we can find complete parametrized solutions by substituting our solutions for s and r.

$a = s^2 - r^2 = n(n - 2m)(4m^2 + 2\mathrm{mn} + n^2)$
$b = 2\mathrm{rs} - s^2 = 4m(m + n)(m - n)^2$
$c = \mathrm{kz} = (4m^4 + 4m^3 n + 6m^2 n^2 - 2\mathrm{mn}^3 + n^4)(2m^2 - 2\mathrm{mn} - n^2)$

Picking values for m and n will give us a solution to $a^3 + b^3 = c^2$

4 Case 2

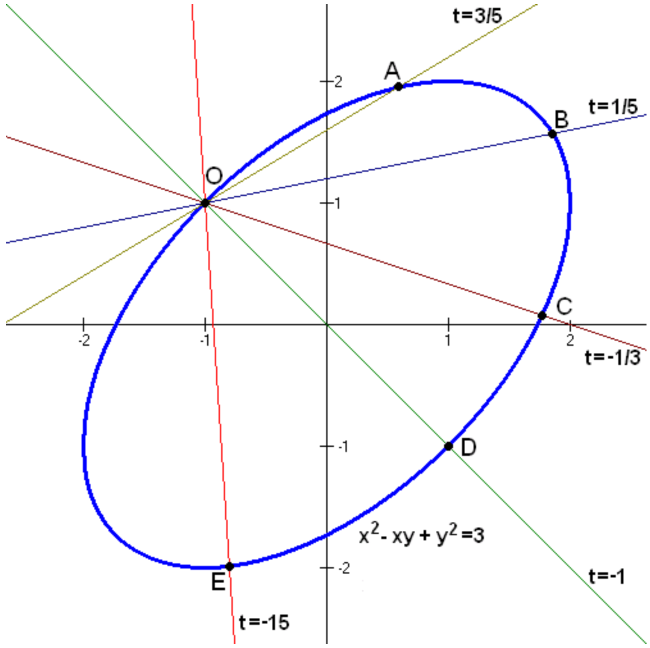
4.1 Parametrization when $\boldsymbol{a^2 - ab + b^2}$ is three times a square

In this case, we set $a^2 - \mathrm{ab} + b^2 = 3d^2$. Next we can say that $x^2 - \mathrm{xy} + y^2 = 3$ by setting $x = \frac{a}{d}$ and $y = \frac{b}{d}$. The point $(-1, 1)$ is a solution to this equation. So $= t(x + 1) + 1$. By substitution, we find:

$$x^2 - x(t(x+1)+1) + (t(x+1))^2 = 3$$
$$(t^2 - t + 1)x^2 + (2t^2 + t - 1) + t^2 - 2 = 0$$
$$(x+1)(x + \frac{t^2-2}{t^2-t+1}) = 0. \text{ So } x = \frac{2-t^2}{t^2-t+1}$$

Now, we find that $y = t(\frac{2-t^2}{t^2-t+1} + 1) + 1 = \frac{2t+1}{t^2-t+1}$

The parametrized solution is then $(x, y) = (\frac{2-t^2}{t^2-t+1}, \frac{2t+1}{t^2-t+1})$



Above is the graph of $x^2 - \mathrm{xy} + y^2 = 3$ with lines of slope t from the point $(-1, 1)$

Slope	Point of Intersection	Coordinates
t		$(\frac{2-t^2}{t^2-t+1}, \frac{2t+1}{t^2-t+1})$
$\frac{3}{5}$	A	$(\frac{41}{19}, \frac{55}{19})$
$\frac{1}{5}$	B	$(\frac{7}{3}, \frac{5}{3})$
$-\frac{1}{3}$	C	$(\frac{17}{13}, \frac{3}{13})$
-1	D	$(\frac{1}{3}, \frac{-1}{3})$
-15	E	$(\frac{-223}{241}, \frac{-29}{241})$

Parametrized solutions for $a, b,$ and d are

$a = 2 - t^2$ $\qquad b = 2t + 1$ $\qquad d = t^2 - t + 1$

To have integer values of a and b, we set $t = \frac{r}{s}$, and multiply through by s^2

So $\quad a = 2s^2 - r^2 \qquad b = 2\mathrm{rs} + s^2 \qquad d = r^2 - \mathrm{rs} + s^2$

4.2 The second parametrization

We know that $a + b$ is also three times a square. We can set $a + b = 3h^2$, or in terms of r and s, $3s^2 + 2\mathrm{rs} - r^2 = 3h^2$. Setting $x = \frac{s}{h}$ and $y = \frac{r}{h}$ we see that

$$3x^2 + 2\mathrm{xy} - y^2 = 3 \qquad (-1, 0) \text{ is a solution}$$
$$y = t(x+1)$$

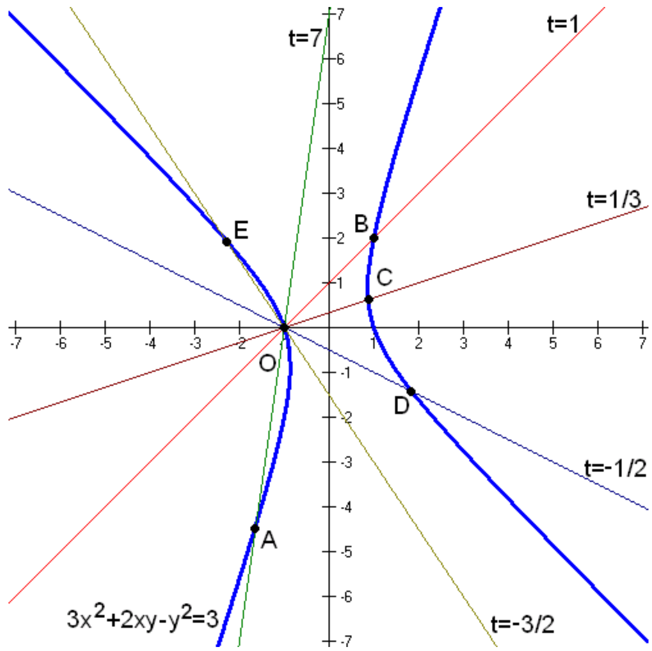
By substitution, we find:

$$3x^2 + 2x^2t + 2\mathrm{xt} - t^2x^2 - 2t^2x - t^2 - 3 = 0$$
$$x^2 + (\frac{2t - t^2}{3 + 2t - t^2})x + \frac{-t^2 - 3}{3 + 2t - t^2} = 0$$

$$(x+1)(x+\frac{-t^2-3}{3+2t-t^2})=0 \qquad x=\frac{t^2+3}{3+2t-t^2}$$

$$y=t(\frac{t^2+3}{3+2t-t^2}+1)=\frac{2t^2+6t}{3+2t-t^2}$$

The parametrized solutions is $(x,y)=(\frac{t^2+3}{3+2t-t^2},\frac{2t^2+6t}{3+2t-t^2})$



Above is the graph of $x^2-\text{xy}+y^2=3$ with lines of slope t from the point $(-1,1)$

Slope	Point of Intersection	Coordinates
t		$(\frac{t^2+3}{3+2t-t^2},\frac{2t^2+6t}{3+2t-t^2})$
7	A	$(\frac{-13}{8},\frac{-35}{8})$
1	B	$(1,2)$
$\frac{1}{3}$	C	$(\frac{7}{8},\frac{5}{8})$
$\frac{-1}{2}$	D	$(\frac{13}{7},\frac{10}{7})$
$\frac{-3}{2}$	E	$(\frac{-7}{3},2)$

So $\quad s=t^2+3 \qquad r=2t^2+6t \qquad h=3+2t-t^2$
To get integer values for s and r we set $t=\frac{m}{n}$ and multiply through by n^2
So $\quad s=m^2+3n^2 \quad r=2m^2+6\text{mn} \qquad h=3n^2+2\text{mn}-m^2$

We find that a and b, in terms of m and n, are
$a=2s^2-r^2=-2m^4-24m^3n-24m^2n^2+18n^4$
$b=2\text{rs}+s^2=(m^2+3n^2)(5m^2+12\text{mn}+3n^2)$
$c=\text{dh}=3(3n^2+2\text{mn}-m^2)(m^4+6m^3n+12m^2n^2-6\text{mn}^3+3n^4)$

5 Conclusion

We now have parametrized solutions for both cases. All we need to do is pick any value for m and n and we can produce a solution to $a^3+b^3=c^2$. This helps illustrate that there exist infinitely many solutions for this equation.

REFERENCES

1- Buddenhagen, Jim. “Can the Cubes of Two Prime Numbers Sum to a Square?” http://home.earthlink.net/˜jbuddenh/num˙theory/sum˙of˙2˙cubes˙a˙square.htm