Harvard Math 129: Algebraic Number Theory
Homework Assignment 5

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Due: Thursday, March 17, 2005

The problems have equal point value, and multi-part problems are of the same value. In any problem where you use a computer, include in your solution the exact commands you type and their output. You may use any software, including (but not limited to) MAGMA and PARI.

1. For each of the following three fields, determining if there is an order of discriminant 20 contained in its ring of integers:

$$K = \mathbb{Q}(\sqrt{5}), \quad K = \mathbb{Q}(\sqrt[3]{2}), \quad \text{and} \ldots$$

   K any extension of $\mathbb{Q}$ of degree 2005. [Hint: for the last one, apply the exact form of our theorem about finiteness of class groups to the unit ideal to show that the discriminant of a degree 2005 field must be large.]

2. Prove that the quantity $C_{r,s}$ in our theorem about finiteness of the class group can be taken to be $\left(\frac{4}{\pi}\right)^s \frac{n!}{n^n}$, as follows (adapted from pg. 19 of Swinnerton-Dyer, "A brief guide to algebraic number theory."): Let S be the set of elements $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that

$$|x_1| + \cdots |x_r| + 2 \sum_{v=r+1}^{r+s} \sqrt{x_v^2 + x_{v+s}^2} \leq 1.$$

   (a) Prove that S is convex and that $M = n^{-n}$, where

$$M = \max\{|x_1 \cdots x_r \cdot (x_{r+1}^2 + x_{(r+1)+s}^2) \cdots (x_{r+s}^2 + x_n^2)| : (x_1, \ldots, x_n) \in S\}.$$

[Hint: For convexity, use the triangle inequality and that for $0 \leq \lambda \leq 1$, we have

$$\lambda\sqrt{x_1^2+y_1^2}+(1-\lambda)\sqrt{x_2^2+y_2^2} \geq \sqrt{(\lambda x_1+(1-\lambda)x_2)^2+(\lambda y_1+(1-\lambda)y_2)^2}$$

for $0 \leq \lambda \leq 1$. In polar coordinates this last inequality is

$$\lambda r_1+(1-\lambda)r_2 \geq \sqrt{\lambda^2 r_1^2+2\lambda(1-\lambda)r_1 r_2\cos(\theta_1-\theta_2)+(1-\lambda)^2 r_2^2},$$

which is trivial. That $M \leq n^{-n}$ follows from the inequality between the arithmetic and geometric means.

(b) Transforming pairs x_v, x_{v+s} from Cartesian to polar coordinates, show also that $v = 2^r(2\pi)^s D_{r,s}(1)$, where

$$D_{\ell,m}(t) = \int\cdots\int_{\mathcal{R}_{\ell,m}(t)} y_1\cdots y_m dx_1\cdots dx_\ell dy_1\cdots dy_m$$

and $\mathcal{R}_{\ell,\Updownarrow}(t)$ is given by $x_\rho \geq 0$ $(1 \leq \rho \leq \ell)$, $y_\rho \geq 0$ $(1 \leq \rho \leq m)$ and

$$x_1+\cdots+x_\ell+2(y_1+\cdots+y_m) \leq t.$$

(c) Prove that

$$D_{\ell,m}(t) = \int_0^t D_{\ell-1,m}(t-x)dx = \int_0^{t/2} D_{\ell,m-1}(t-2y)ydy$$

and deduce by induction that

$$D_{\ell,m}(t) = \frac{4^{-m}t^{\ell+2m}}{(\ell+2m)!}$$